MATH 165 - Linear Algebra w/ differential Equations

First Order Differentiable Equations:

What is a differential equation?

↳ Any equation involving one or more derivatives of unknown functions → $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y^2 = 5\sin x$

↳ This is an ordinary differential equation (ODE) meaning it has only one independent variable (x)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$ → Since this equation has multiple variables it is a Partial Differential Equations (PDE)

1.2 Linear Differential Equations

↳ If every term of the DE is a function of x then it is linear

Linear

$y''' + e^{3x}y'' + x^3y' + \cos x(y) = \ln x$

↳ 3rd order

$xy' + \frac{2y}{1+x^2} = 0$ → 1st order

Non-Linear

$y'' + x^4\cos(y') - xy = e^{x^2}$

↳ not a function of x!

$y'' + y^2 = 0$

↳ not a function of x!.

Solving Differential Equations:

↳ any function $f(x) = y$ for all derivatives dictated by the order is a solution of that differential equation

Example: Verify that $x^2 + y^2 - 4 = 0$ implicitly defines a solution to the following nonlinear DE → $\frac{dy}{dx} = \frac{-x}{y}$

↳ differentiate the solution implicitly with respect to x → $2x + 2y\frac{dy}{dx} = 0$

↳ this can be simplified to $dy/dx = -x/y$ ✓

If we define y implicitly in terms of x → $y = \pm\sqrt{4 - x^2}$

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Consider this DiyA: $\frac{dy}{dx} = \frac{y^2}{x^2+1}$

1. Find the unique equilibrium solution

↳ $dy = y^2 \rightarrow y = y^3/3$

$dx = x^2+1 \rightarrow x^3/3 + x$

$y^3 = x^3 + 3x$

$x(x+3)$

↳ $dy(x^2+1) = dx(y^2)$

$\frac{dy(x^2+1)}{dx} = y^2$

Example #2: Verify that $\sin(xy) + y^2 - x = 0$ defines a solution to $\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}$ → differentiate the solution (implicitly) with respect to x

↳ $\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} - 1 = 0$

↳ $\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}$

The General Solution to any n'th degree differential Equation is any solution with n constants that, when defined, produce all possible solutions to the given DE

↳ note: not all DEs have a general solution

Initial Value Problems:

↳ Since general solutions are often too abstract to be useful Initial value problems provide the additional constraints by defining values for the function & its derivatives when $x = 0$ (or whatever the initial value of the independent variable is)

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Initial Value Problem Example #1: Find the solution of the DE
$y'' = 18\cos(3x)$ given that $y(0)=1$ & $y'(0)=4$

↳ Start by finding the general solution

↳ $\boxed{y(x) = -2\cos(3x) + C_1x + C_2}$

Thus, we know that $y(0)=1$ meaning when $x=0$ $1 = -2 + C_2$

↳ $\boxed{C_2 = 3}$

Then since $y'(0)=4 \rightarrow 6\sin(3(0)) \boxed{C_1 = 4}$

Putting these together

The solution is: $y(x) = -2\cos(3x) + 4x + 3$ ✓

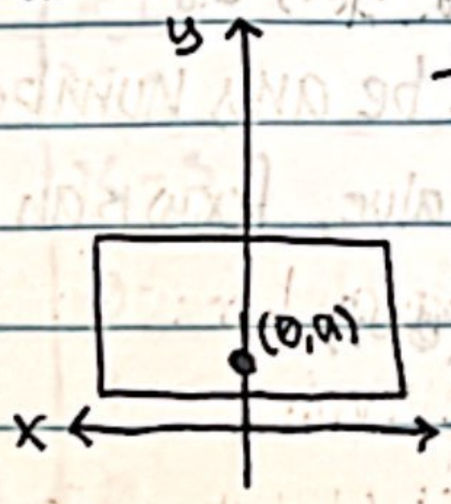
1.3 Geometry of First-Order Differential Equations

Existance & Uniqueness Theorem: ← existance

1. Does the initial-value problem have **any** solutions?
2. Does the initial-value problem have only **one** solution? ← uniqueness

The following differential equation has a unique solution when $a \neq 0$

$\frac{dy}{dx} = 3xy^{\frac{1}{3}}$ & $y(0)=a \rightarrow \frac{\partial f}{\partial y} = xy^{-2/3} \rightarrow$ R is all real numbers except $y=0$



→ the E&U Theorem demands that $\partial f/\partial y$ is continuous and that $dy/dx \neq f(x,y)$

↳ Thus, when $y(0)=0$ this DE does not have a unique solution!

What can we do when a differential equation has more than one solution at a given initial-value?

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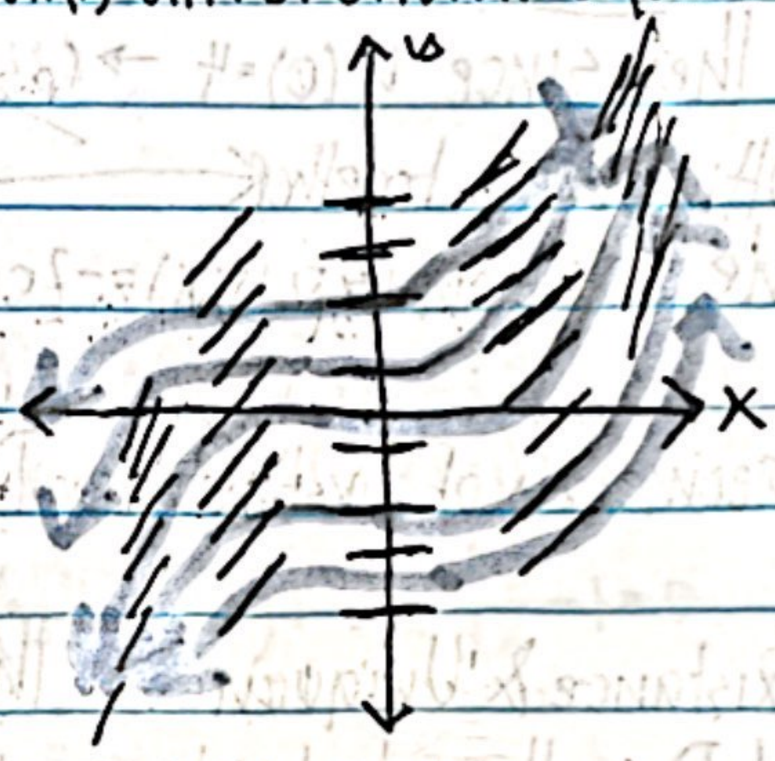
Slope Fields: When $\frac{dy}{dx} = f(x,y)$ the slope of the tangent line is given by $f(x,y)$

→ If we compute & plot $f(x,y)$ at several points we can estimate the general shape of the solution curve

Sketch the slope field for the following differential equation

$\frac{dy}{dx} = 2x^2$

	x	Slope $(2x^2)$
	0	$2(0)^2 = 0$
±	0.2	$2(0.2)^2 = 0.08$
±	0.4	$2(0.4)^2 = 0.32$
±	0.6	$2(0.6)^2 = 0.72$
±	0.8	$2(0.8)^2 = 1.28$
±	1.0	$2(1)^2 = 2$



$y(x) = 2/3\,x^3 + C$ → The blue lines are the <u>isolines</u> of the DE
They are all parallel to each other (= slope)

Equilibrium Solutions: any solution that satisfies $y(x) = y_0$

↳ Recall the initial value is usually 0 but it can be any number

so for equations where $f(x,y) =$ any initial value, $f(x,y)$ is an equilibrium

↳ ex: the equilibrium solution of $\frac{dy}{dx} = (y-x)(y+1)$ is $y(x) = -1$

Concavity Changes: Recall that the 2nd derivative of a function at a point tells us about the functions concavity at that point

↳ By this method we can differentiate any first order DE to get $\frac{d^2y}{d^2x}$ which can tell us about the concavity of the solution curve

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Slope Field Example #2: Sketch the slope field of the following DE:

$\frac{dy}{dx} = y(2-y)$

1. Equilibrium Solutions:

$y(x)=0$ & $y(x)=2$

Thus, by the E&U theorem

$R=\{y \mid y<0,\ 0<y<2,\ y>2\}$

dy/dx: $\underset{-}{\downarrow}$ $\underset{+}{\downarrow}$ $\underset{-}{\downarrow}$

2. Isolines: $y(2-y)=k \rightarrow y^2-2y+k=0$

↳ The solution curves have a slope of k whenever they intersect with $y=1\pm\sqrt{1-k}$

3. Concavity: take the derivative of dy/dx

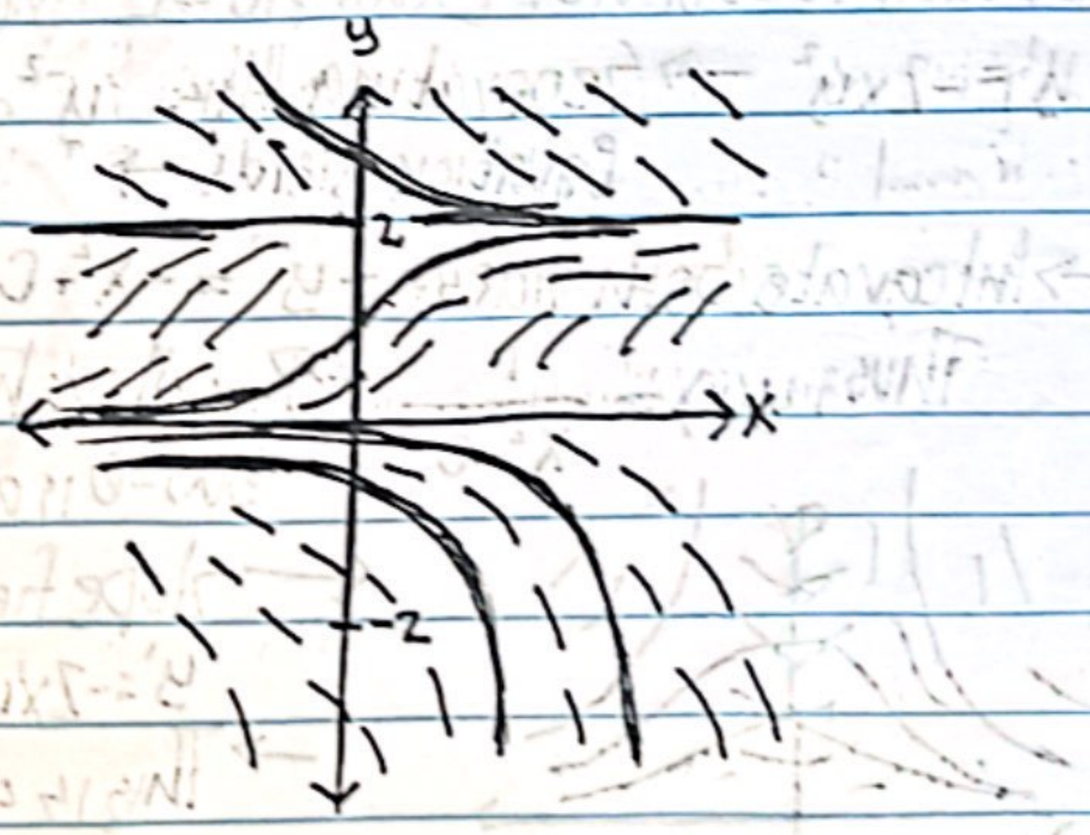
$\frac{d^2y}{d^2x} = 2y(1-y)(2-y) \rightarrow 0, 1, \& 2$

$y<0,\ 0<y<1,\ 1<y<2,\ 2<y$

$\frac{d^2y}{d^2x}$ $\underset{-}{\downarrow}$ $\underset{+}{\downarrow}$ $\underset{-}{\downarrow}$ $\underset{+}{\downarrow}$

Thus we can plug values into dy/dx & graph

k	slope of sol curves
-3	~~-15~~
-2	~~-8~~
-1	-5
0	0
1	1
2	0
3	-3



$$y(x)=\frac{2ce^{2x}}{ce^{2x}-1}$$

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1.4 Seperable Differential Equations

↳ If a DE can be written in the following form it is **seperable**

$P(y)\frac{dy}{dx} = q(x)$ → If $P(y)$ & $q(x)$ are continious then the general solution is given by:

$$\int P(y)\,dy = \int q(x)\,dx + C$$

↳ Using the chain rule this formula can be rewritten

$\frac{d}{dx}\left(\int P(y)\,dy\right) = q(x)$ → Integrating both sides of this equation yields the eq. above

Seperable Differential Equations Example #1:

Solve $(e^y + 4y^2)\frac{dy}{dx} = 9xe^{3x}$ → simply apply the above formula!

$$\int P(y)\,dy = \int q(x)\,dx + C \rightarrow \int (e^y + 4y^2)\,dy = \int 9xe^{3x}\,dx + C$$

(Integration by parts!)

$$\rightarrow e^y + \frac{4}{3}y^3 = 3xe^{3x} - e^{3x} + C \rightarrow \boxed{3e^y + 4y^3 = 3e^{3x}(3x-1) + 3C}$$

Let's amp it up a bit! Example #2 Find all solutions to

$y' = -2xy^2$ → Seperating this equation yields → $y^{-2}\,dy = -2x\,dx$

↳ Integrate both sides: $-y^{-1} = -x^2 + C$

Thus: $y(x) = \frac{1}{x^2 - C}$ → Recall we had to divide by y so $y(x) = 0$ is also a solution!

← slope field & sol. curves of $y' = -2xy^2$

→ This is so fucked dude...

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1.6 First-Order Linear Differential Equations ←

↳ If a DE can be written in the following form it is FOL

$a(x)\frac{dy}{dx} + b(x)y = r(x)$ → $a \neq 0$ on interval (a,b)

OR

Standard Form: $\frac{dy}{dx} + P(x)y = g(x)$ Where: $P(x) = b(x)/a(x)$, $g(x) = r(x)/a(x)$

We are trying to get the DE into the form of:

$\frac{d}{dx}[g(x,y)] = F(x)$ → Where the solution can be derived & Integrated from $F(x)$

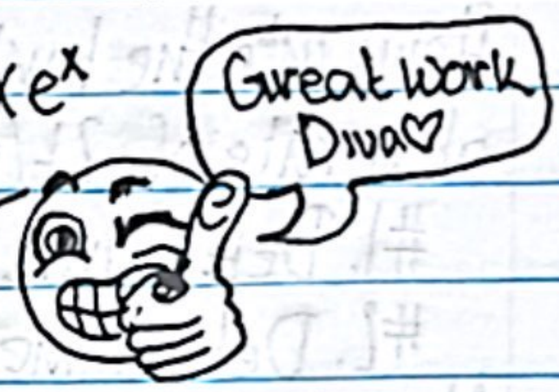
Example #1: Solve $\frac{dy}{dx} + \frac{1}{x}y = e^x$, Where $x > 0$

$x \cdot \left(\frac{dy}{dx} + \frac{1}{x}y\right) = xe^x \rightarrow x\frac{dy}{dx} + y = xe^x$ → This is so chain-rule core

$\frac{d}{dx}(xy) = xe^x$

Integrate both sides! ←

$xy = xe^x - e^x + C \rightarrow \boxed{y(x) = x^{-1}(e^x(x-1) + C)}$



Example #2: Consider the following DE & Initial value

$\frac{dy}{dx} + xy = xe^{x^2/2}$ When $y(0) = 1$

→ To solve establish an Integrating factor

$I(x) = e^{\int x\,dx} = e^{x^2/2}$

→ Then multiply both sides of the DE by this factor

$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2}$

↳ $e^{x^2/2}y = \frac{1}{2}e^{x^2} + C$ → Solving for C when $y(0) = 1$ gives

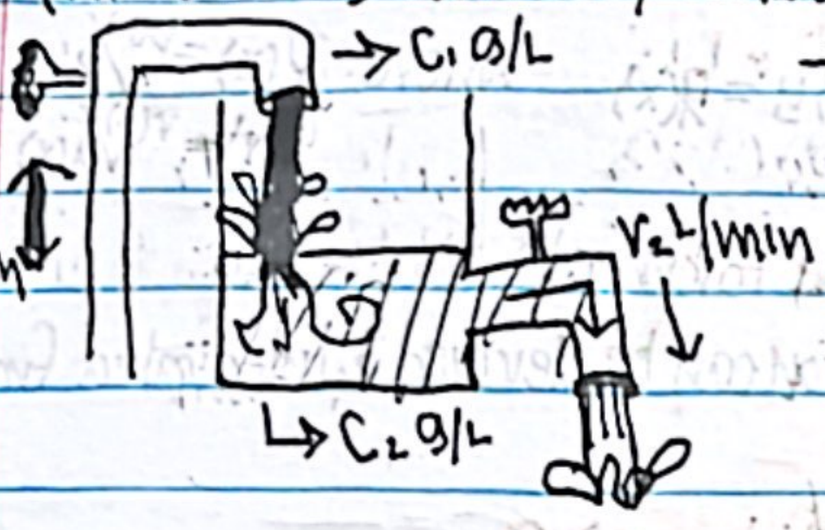
$1 = \frac{1}{2} + C \rightarrow C = 1/2$

Thus: $\boxed{y(x) = \frac{1}{2}e^{-x^2/2}(e^{x^2} + 1)}$

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1.7 Applications of First-Order Linear Differential Equations

Mixing Problems: Consider two distinct liquids, one being added into the same container as the other at a constant rate (while being stirred), while being drained at a constant rate

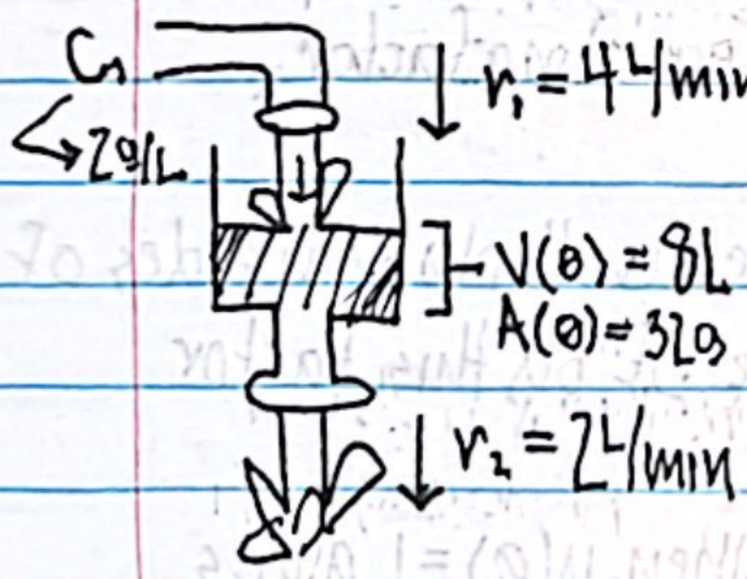


→ A(t) = amount of chemical in the tank at t
V(t) = Volume of solution at time t
& $c_2 = A(t)/V(t)$ = Concentration at t
→ We will explain how to express this kind of situation as a FOLDE

Example: A tank contains 8L of water in which 32 grams of Cuntium is dizzolved. A solution containing 2g/L of the chemcal flows into the tank at a rate of 4 L/min & the mixture is drained at a rate of 2 L/min

#1. Determine the amount of chemical in the tank after 20min
#2. Determine the concentration in the tank at that time

Alright, there is a lot of info here, lets draw a picture & assing our given values to it



→ #1 is asking us to find A(20)
#2 is asking us to find A(20)/V(20)

The change in volume is given bus

$$\Delta V = r_1 \Delta t - r_2 \Delta t$$

Which gives us $dV/dt = 2$

→ Integrating dV/dt with the constraint of $V(0) = 8$ gives us

$$V(t) = 2(t+4)$$

The change in the amout of chemical is given by:

$$\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t \rightarrow dA/dt = 8 - 2c_2$$

↳ Recall $c_2 = A/V \rightarrow \frac{dA}{dt} + \frac{1}{t+4} A = 8$

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Applications of FOLDEs Cont.

Notice we now ave left with a linear DE:

→ $\frac{dy}{dx}+P(x)y=q(x) \rightarrow \frac{dA}{dt}+\frac{1}{t+4}A=8$

Find the Integrating factor: $1=e^{\int \frac{1}{t+4}dt}=t+4$

↳ Thus, $\frac{d}{dt}((t+4)A)=8(t+4)$ → Integrate both sides! → $(t+4)A=4(t+4)^2+C$

↳ When $A(8)=32$, $C=64$

Finally: #1: $A(20)=\frac{1}{6}(24^2+16)=296/3$ g

#2: $\frac{A(20)}{V(20)}=\frac{1}{48}\cdot\frac{296}{3}=37/18$ g/L

Electrical Circuits

The most basic electrical circut is made by connecting a wire to the terminals of a battery this causes a flow of charge ($q(t)$)

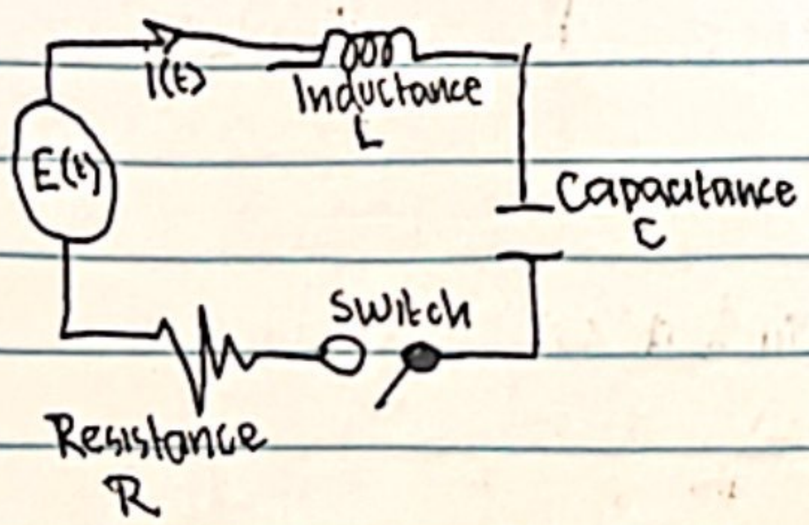
↳ measured in coulombs (C)

This produces the current described by $i(t)$

↳ measured in Amperes (A) → $i(t)=\frac{dq}{dt}$

Kirchhoff's Second Law: the sum of the voltage drops around a closed circut is zero → What are the components of a closed circut?

1. Resistors: resists the flow of the charge
2. Capacitors: stores charge, thus obstructing flow
3. Inductors: Opposes any change in the current running through it
4. EMF: the source of the voltage that produces an electromagnetic force



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Application: Newton's Law of Cooling:
An object whose temperature is 350°F is placed in a room that is 70°F. After 2 minutes the object is 210°F
What is the temp of the object after 4 minutes?
↳ Newton's 2nd law of Cooling:

$\frac{dT}{dt} = -K(T - T_m)$ where temp at time t is minutes & T_m is the temp of the room

$T_m = 70°F \quad T(0) = 350 \quad T(2) = 210$, Thus:

$\frac{dT}{dt} = -K(T-70)$ → Seperate the variables:

$\frac{1}{T-70}\,dT = -K\,dt$

$\int \frac{1}{T-70}\,dT = \int -K\,dt \rightarrow \ln(|T-70|) = -kt \rightarrow 70 + Ce^{-kt}$

→ Since $T(0) = 350$

↳ ~~$|T-70| = Ce^{-kt}$~~ $T = 70 + Ce^{-kt}$

$350 = 70 + C$

$C = 280$

Thus, $T = 70(1 + 4e^{-kt})$

$210 = 70(1 + 4e^{-2k}) \rightarrow K = \frac{1}{2}\ln 2$

$T = 70(1 + 4e^{-t/2(\ln 2)})$

$T(4) = 70(1 + 4e^{-4\ln 2}) = \boxed{140°\,F}$

$T(0) = 350$ find $T(4)$
$T(2) = 210$
$T_m = 70$

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2.1 Matricies: Definition, Notation & Syntax

When considering systems of linear equations matrix representa
will be useful: Consider the following system of equation

$$\begin{aligned} 5x_1 - x_2 - 3x_3 &= -1 \\ -x_1 + 4x_2 - 8x_3 &= 2 \\ 6x_1 + 8x_3 &= -5 \end{aligned} \longrightarrow \begin{bmatrix} 5 & -1 & -3 & -1 \\ -1 & 4 & -8 & 2 \\ 6 & 0 & 8 & -5 \end{bmatrix}$$

By representing the system as a matrix it allows it to be solved using matrix algebra. Eventually allowing us to solve linear DEs too

The basics of matrices:

- ↳ a matrix is simply an array of numbers
- ↳ Entries in a matrix are called elements

Two matricies, A and B are equal if

1. They are the same size, $m \times n$
2. The corresponding elements are equal

We can find the transpose of a matrix by swapping the rows & columns

Square matricies have the same number of rows & columns

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2.2 Introduction to Matrix Algebra

↳ You can only add & subtract matrices of the same dimensions

Multiplying a matrix by a column:

$$\begin{bmatrix} -3 & 1 & -2 \\ 0 & 5 & -2 \\ -4 & -2 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} (-3)(-2)+(1)(1)+(-2)(6) \\ (0)(-2)+(5)(1)+(-2)(6) \\ (-4)(-2)+(-2)(1)+(5)(6) \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 36 \end{bmatrix}$$

Multiplying matrix $m \times n$ by matrix $n \times m$

$$\begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 3 & -1 \\ -9 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \\ -9 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -16 & 3 \\ -76 & 18 \end{bmatrix}$$

Or what about this:

$$\begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}[1,-3] = \left[\begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}[1], \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}[-3]\right] = \begin{bmatrix} 6 & -18 \\ -4 & 12 \\ 0 & 0 \\ 1 & -3 \end{bmatrix}$$

⚠ Matrix multiplication is NOT commutative! ⚠

Each matrix also has a corresponding Identity Matrix (I_n)

↳ Which is a $n \times n$ square matrix with ones in the main diagonal and all other elements are zero

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The elements of an Identity matrix can be described by the Kronecker Delta symbol → $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ → Thus: $I_n = [\delta_{ij}]$

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Properties of Identity Matrices:

↳ $A_{m\times n}I_n = A_{m\times n}$ → Example: Let $A = \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix}$

↳ $I_m A_{m\times p} = A_{m\times p}$

1. $A_{m\times n}I_n = AI_3 = \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$

2. $I_m A_{m\times p} - I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix} = A$

Properties of Transpose Matricies:

1. $(A^T)^T = A$
2. $(A+C)^T = A^T + C^T$
3. $(AB)^T = B^T A^T$

When performing Calculus on Matrices the order of multiplycation must be preserved

2.3 Additional Terminology for Systems of Linear Equations

Homogenous systems: if all equations in a system are equal to zero then it is homogenous otherwise its nonhomogenous

The solution to a system of linear equations is an ordered list of x-values that satisfy the system

Example: Verify that for all real numbers a & b $(25+5a-10b, -8-4a+3b, a, b)$ satisfies the following system

$x_1 + 2x_2 + 3x_3 + 4x_4 = 8$

$2x_1 + 5x_2 + 10x_3 + 5x_4 = 8$

To verify that the given list is a solution plug the values in sequentially

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FOLDE & Initial Value Review:

~~Recall a FOLDE~~ ~~Recall the definition of FOLDE:~~

1.4 Seperable DEs: Recall: If a DE can be written in the form:

$\frac{dy}{dx} = P(x)q(y)$ If $y \neq 0$, then $\frac{1}{q(y)} y' = P(x)$

↳ Integrating this yields: $\int \frac{1}{q(y)} y' \, dx = \int P(x) dx$

↳ Then by u-substitution yields: $u = y(x) \rightarrow dx = \frac{1}{y'} du$

$$N(y) = \int \frac{1}{q(u)} du = \int P(x) dx$$ * assuming $\frac{1}{q}$ is continious

↳ Thus yielding: $N(y) = \int P(x) dx + C$

Which can be used for y

↳ Sometimes we can solve for an explicit solution $y(x)$

Otherwise, the equation still gives an implicit solution

↳ ie: still defines the relationship between x & y

↳ Example: Newtons 2nd Law of Cooling:

$\frac{dT}{dt} = -k(T-m)$ * Where K & M are constants

$\int \frac{1}{T-m} dT = \int -k dt + C \rightarrow \ln|T-m| = -kt + C$

↳ $|T-m| = e^{-kt+C}$

What about $T(0) = 100$ & $m = 50$

$|T-m| = e^{C} e^{-kt}$ ↳ new constant!

solving for the constat: $|T-m| = C_1 e^{-kt}$

$50 = Ce^{0} \rightarrow C = 50$

Finally, we have a general solution

Thus, $T(t) = 50 + 50e^{-kt}$

$T = m + Ce^{-kt}$

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1.6: First Order Linear Differential Equations:

$a_1(x)y' + a_0(x)y = F(x)$

↳ Which can be rewritten:

$y' + P(x)y = q(x)$

If $a_1(x) \neq 0$, we can write:

$$y' + \frac{a_0(x)}{a_1(x)} = \frac{F(x)}{a_1(x)}$$

Integrating Factors method:

↳ Multiply both sides of this FOLDE by an Integrating Factor

$I(x)y' + P(x)I(x)y = q(x)I(x)$ let $\frac{dI}{dx} = P(x)I(x)$

Then, by the product rule:

→ $\frac{d}{dx}(I(x)y) = I(x)y' + P(x)I(x)y = q(x)I(x)$

→ $I(x)y = \int q(x)I(x)dx + C$

→ Solving for y gives: $y = \frac{1}{I(x)}\int q(x)I(x)dx + \frac{C}{I(x)}$

If we can find a function $I(x)$ such that:

$\frac{dI}{dx} = P(x)I(x) = P(x)I$ → It becomes a seperable DE!

↳ $\int \frac{1}{I} dI = \int P(x)dx + C = \ln|I| = \int P(x)dx + C$

↳ $I = e^{\int Pxdx + C}$ ↔ $I = C_1 e^{\int Pxdx}$ → Since we can choose any $I(x)$ we can let $C=1$

$I(x) = e^{\int Pxdx}$

In Summary: A FOLDE of the form $y' + P(x)y = q(x)$ then the genral solution is given by

$y(x) = \frac{1}{I(x)}\left(\int I(x)q(x)dx + C\right)$ where $I(x) = e^{\int P(x)dx}$

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Integrating Factor Example:

$xy' - xy = 3x$ when $x > 0$

↳ Start by putting the DE into standard form

$(y' + P(x)y = q(x)) \rightarrow P(x) = -1 \quad q(x) = 3$

$$I(x) = e^{\int P(x)dx} = e^{\int -1 dx} = e^{-x}$$

$$y(x) = \frac{1}{I(x)}\left(\int I(x)q(x)dx + C\right) = e^{x}\left(\int 3e^{-x}dx + C\right) = (-3e^{-x} + C)e^{x}$$

$$= \boxed{-3 + Ce^{x}} \rightarrow \text{General Solution}$$

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More Matrix Notation

$A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}$!! Matrix multiplication is **not** commutative!!
meaning $AB \neq BA$

$$AB=\begin{bmatrix}1(0)+2(1) & 1(-2)+2(-1)\\ -1(0)+0(1) & (-1)(-2)+0(-1)\end{bmatrix}=\begin{bmatrix}2 & -4\\ 0 & 2\end{bmatrix}$$

$$BA=\begin{bmatrix}0(1)+-2(-1) & 0(2)+-2(0)\\ (1)(1)+(-1)(-1) & 1(2)+-1(0)\end{bmatrix}=\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}$$

Color Coded Matrix Multiplication Example:

$A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$ $B=\begin{bmatrix}0 & 4\\ 3 & 2\end{bmatrix}$ Find AB

↳ $AB=\begin{bmatrix}1(0)+2(3) & 1(4)+2(2)\\ -1(0)+3(3) & -1(4)+3(2)\end{bmatrix}=\begin{bmatrix}6 & 8\\ 9 & 2\end{bmatrix}$ → Lets make this worse!

$A=\begin{bmatrix}1 & 0 & -1 & 2\\ 1 & 0 & 2 & 2\end{bmatrix}$ $B=\begin{bmatrix}2 & -3 & 1\\ 0 & 0 & 0\\ 4 & 6 & 1\\ 5 & 1 & 2\end{bmatrix}$ Find AB

↳ $(2\times 4)(4\times 3)=(2\times 3)$

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49+24-42+9

More Complicated Matrix Multiplication

$$A=\begin{bmatrix}1&0&-1&2\\1&0&2&2\end{bmatrix} \qquad B=\begin{bmatrix}2&-3&1\\0&0&0\\4&6&1\\5&1&2\end{bmatrix}$$

Find AB:

$(2\times 4)(4\times 3)=(2\times 3)$

$$AB=\begin{bmatrix}1(2)+0(0)+-1(4)+2(5) & 1(-3)+0(0)+-1(6)+2(1) & 1(1)+0(0)+-1(1)+2(2)\\ 1(2)+0(0)+0(4)+2(5) & 1(-3)+0(0)+2(6)+2(1) & 1(1)+0(0)+2(1)+2(2)\end{bmatrix}$$

$$=\begin{bmatrix}8&-7&4\\20&11&7\end{bmatrix}$$

6

Matrix Vocab

$\begin{bmatrix}1&2&3\\2&4&6\\3&3&3\end{bmatrix}$ → Main diagonal

$\begin{bmatrix}3&1&6\\0&2&9\\0&0&3\end{bmatrix}$ → upper triangle matrix

$\begin{bmatrix}1&0&0\\2&4&0\\3&9&1\end{bmatrix}$ → lower triangle matrix

$\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$ → diagonal matrix

The transpose of a upper triangular matrix is a lower triangular matrix

The product of two upper triangle matrixies is an upper triangle matrix (same w/ lower)

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2.3: System of Linear Equations

Example: $3x_1 + 2x_2 = 4$

$x_1 - 5x_2 = 3$

↳ The corresponding coefficient matrix is given by:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix} \quad \& \quad A^{\#} = \left[\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -5 & 3 \end{array}\right]$$

We will often switch from equations to matricies & vice versa

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ x + y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Dilema: $\underbrace{\begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow A\vec{X} = \vec{B}$

$\vec{X} = A^{-1}\vec{B}$

↑ When does A^{-1} exist?

↑ System constants

↑ unknowns

2 New Strategies for solving:

1. Back Substitution
2. Elimination of Variables

Generally, we will perform elimination until the system becomes simple enough to use substitution

Example: $x_1 + x_2 + x_3 = 6$
$x_2 - 4x_3 = -4$
$x_3 = 3$

→ This system shows the general structure for back substitution

↳ We can put this system into rowechelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Notice the first entry in each row is 1

↳ If any of the leading coefs $\neq 1$ then the matrix is not in REF

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Elementary Row Operations:

↳ These are the modifications we can make to a system to find the solution set

1. Multiply by a non-zero constant
2. Permutate any 2 equations
3. Add a constant multiple of an equation to any other equation

Gaussian Elimination:

1. Convert to Auamented Matrix
2. Using EROs find a row-equivilent Aug. Matrix in REF
3. Convert the REF back into a system of equations
4. Solve the system using back substitution

Example: $3x-2y=4$, $x+y=1$ $\rightarrow \left[\begin{array}{cc|c} 3 & -2 & 4 \\ 1 & 1 & 1 \end{array}\right]$ ① convert to Aug Matrix

② Put in REF $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 1 & 1 & 1 \end{array}\right]$ $R_1 \rightarrow R_2$, $R_2 \rightarrow R_1$ $\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & -2 & 4 \end{array}\right]$

$-3R_1+R_2 \rightarrow R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & 1 \end{array}\right]$

$-1/5 R_2 \rightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1/5 \end{array}\right] \rightarrow$ REF ✓

③ REF → SOLE: $x+y=1$, $y=-1/5$ ④ Solve!

$\boxed{x=6/5 \text{ \& } y=-1/5}$

Example #2:

$2x_1+x_3=1$

$x_1+2x_2+x_3=2$

$x_2=1$

① $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$ $R_2 \rightarrow R_1$, $R_1 \rightarrow R_2$ = ② $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$

$-2R_1+R_2 \rightarrow R_2 = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -4 & -1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right]$ $R_2 \rightarrow R_3$, $R_3 \rightarrow R_2$ $= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -4 & -1 & -3 \end{array}\right]$

→

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Example #2 cont…

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -4 & 1 & -3 \end{array}\right] \; 4R_2+R_3 \to R_3 \hookrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \; -R_3 \to R_3 \hookrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$ REF ✓

(3.) $x_1 + 2x_2 + x_3 = 2$

$x_2 = 1$

$x_3 = -1$

(4.) Solve!

$x_1 + 2 - 1 = 2$

$x_1 = 1$

$\rightarrow \boxed{\{(1, 1, -1)\}}$

Example #3:

$$\left[\begin{array}{cc} 2 & 3 \\ 1 & 4 \end{array}\right] \begin{array}{l} R_1 \to R_2 \\ R_2 \to R_1 \end{array} = \left[\begin{array}{cc} 1 & 4 \\ 2 & 3 \end{array}\right] \; -2R_1+R_2 \to R_2 = \left[\begin{array}{cc} 1 & 4 \\ 0 & -5 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc} 1 & 4 \\ 0 & 1 \end{array}\right]$$

↳ This can be further simplified into Reduced REF by "clearing" the 4

$$\left[\begin{array}{cc} 1 & 4 \\ 0 & 1 \end{array}\right] \xrightarrow{-4R_2+R_1 \to R_1} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

Note: every RREF is also a REF

Example #4:

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & -1 \\ 1 & -1 & 5 & 0 \\ 2 & 2 & 10 & -4 \end{array}\right] \begin{array}{l} R_2 \to R_1 \\ R_1 \to R_3 \\ R_3 \to R_2 \end{array} = \left[\begin{array}{cccc} 1 & -1 & 5 & 0 \\ 2 & 2 & 10 & -4 \\ 0 & 1 & 0 & -1 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_3} \left[\begin{array}{cccc} 1 & -1 & 5 & 0 \\ 0 & 4 & 0 & -4 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{cccc} 1 & -1 & 5 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{cccc} 1 & -1 & 5 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ REF ✓} \xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{cccc} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right] \text{ RREF ✓}$$

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Example #5: Gaussian Elimination, No Solution

$x_1 + x_2 - x_3 + x_4 = 1$
$2x_1 + 3x_2 + x_3 = 4$
$3x_1 + 5x_2 + 3x_3 - x_4 = 5$

① Convert to Aug. Matrix:

$$A = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right]$$

② Convert to REF:

$$\begin{matrix} -2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \end{matrix} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$-R_2+R_3 \to R_3 \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

→ The last row of this matrix is equivilant to $0 = -1$ meaning there is no solution!

Example #6: Gaussian Elimination, Infinitely Many Solutions

$3x_1 - x_2 = 1$
$2x_1 + x_2 + 5x_3 = 4$
$7x_1 - 5x_2 - 8x_3 = -3$

① Convert to Aug. Matrix

$$A = \left[\begin{array}{ccc|c} 3 & -1 & 0 & 1 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array}\right] \quad -R_2+R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array}\right]$$

$$\begin{matrix} -2R_1+R_2 \to R_2 \\ -7R_1+R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 5 & 15 & 10 \\ 0 & 9 & 27 & 18 \end{array}\right] \quad \begin{matrix} \frac{1}{5}R_2 \to R_2 \\ \frac{1}{4}R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$-R_2+R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 2R_2+R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last row is all 0 it is a free variable meaning that it is satisfied by

The Solution to this system is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$ for any & all $t \in \mathbb{R}$

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Gauss-Jordan Elimination:

↳ This method of solving systems of Linear Equations relies on putting a matrix into RREF

↳ It gives an easier to read solution set but requires more EROs

Consider the following System:

$$\begin{aligned} 4x_1 - 3x_2 + 6x_3 &= 2 \\ x_1 - 3x_2 + 6x_3 &= 5 \\ 2x_1 + 3x_2 - 8x_3 &= -6 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & -6 \end{array}\right] \xrightarrow{\substack{R_1 \to R_2 \\ R_2 \to R_1}} \left[\begin{array}{ccc|c} -1 & -3 & 6 & 5 \\ 4 & -3 & 6 & 2 \\ -2 & 3 & -8 & -6 \end{array}\right]$$

$$\xrightarrow{\substack{-4R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3}} \left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 0 & 9 & -18 & -18 \\ 0 & -3 & 4 & 4 \end{array}\right] \xrightarrow{\frac{1}{9}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & -3 & 4 & 4 \end{array}\right] \xrightarrow{3R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{2R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{RREF!}$$

Reduced Row Echelon Form (RREF): Any $n \times n$ matrix in which the leading term is 1 with zeros above & below the leading 1s.

$$A = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 \\ 1 & 4 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{RREF} \checkmark \qquad B = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \qquad C = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \times$$

A&B are in RREF, but C is not because the leading term of $R_1(C)$ is -1 & in $R_2(C)$, $R_1(C)$ there are nonzero elements above the leading term of $R_3(C)$

The **trace** of a matrix is the sum of the elements of the main diagonal (denoted trace(M) or tr(M))

$tr(A) = 5$

$trace(B) = 3$

$tr(C) = 0$

Any matrix $n \times n$ with $tr(M) = 0$ is a linear combo of the commuters of pairs of matrices

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2.6 Inverse of Square Matricies:

↳ For a given $n \times n$ matrix A, there exists a matrix B that satisfies $AB = I_n$ & $BA = I_n$

↳ In the case that such a matrix B exists for a linear system it can be written as $Ax = b$

$(BA)x = Bb$ if $BA = I_n$ then $\rightarrow$ $x = Bb$

Let A be an $n \times n$ matrix. Suppose B & C are also $n \times n$ matricies satisfies $AB = BA = I_n$ Then, $B = C$

$AC = CA = I_n$

↳ If there exists a matrix A^{-1} such that

$AA^{-1} = A^{-1}A = I_n$

A^{-1} is called the inverse of A

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Verify that B is the inverse of A

↳ multiplying AB and BA both equal $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If A^{-1} exists then the $n \times n$ system $Ax = b$ has the unique solution $x = A^{-1}b$

Note on RREF: each matrix has exactly one RREF

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Example of Gausse-Jordan Elimination

Find A^{-1} and solve

$$A=\begin{bmatrix}1&1&3\\0&1&2\\3&5&-1\end{bmatrix}\rightarrow\left[\begin{array}{ccc|ccc}1&1&3&1&0&0\\0&1&2&0&1&0\\3&5&-1&0&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&1&3&1&0&0\\0&1&2&0&1&0\\0&2&-10&-3&0&1\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&1&1&-1&0\\0&1&2&0&1&0\\0&0&-14&-3&-2&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&1&1&-1&0\\0&1&2&0&1&0\\0&0&1&3/14&1/7&-1/14\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&11/14&-8/7&1/14\\0&1&0&-3/7&5/7&1/7\\0&0&1&3/14&1/7&-1/14\end{array}\right]\qquad A^{-1}=\begin{bmatrix}11/14&-8/7&1/14\\-3/7&5/7&1/7\\3/14&1/7&-1/14\end{bmatrix}$$

↳ When performing these EROs treat the right side of the matrix as three distinct single columns.

Now, using this shiny new inverse matrix solve

$$x_1+x_2+3x_3=2$$
$$x_2+2x_3=1$$
$$3x_1+5x_2-x_3=4$$

Rewrite $Ax=b\rightarrow b=\begin{bmatrix}2\\1\\4\end{bmatrix}$

Recall: an invertable system has a unique solution

When $X=A^{-1}b$

$$X=\begin{bmatrix}11/14&-8/7&1/14\\-3/7&5/7&1/7\\3/14&1/7&-1/14\end{bmatrix}\begin{bmatrix}2\\1\\4\end{bmatrix}=\begin{bmatrix}5/7\\3/7\\2/7\end{bmatrix}$$

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3.1 Introduction to Determinants

↳ Recall: not every matrix has an inverese
How do we tell if a matrix has one?
The Determinant!

In 2×2 Matricies:

$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ → A 2×2 matrix only has an inverse if its rank is 2

To put A in REF $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}-\frac{a_{12}a_{21}}{a_{11}} \end{bmatrix}$ In order for a to be invertable

$a_{22}-\frac{a_{12}a_{21}}{a_{11}} \neq 0$

For 2×2 matricies

$\det(A)=a_{11}a_{22}-a_{12}a_{21}$

The rank of a matrix is the number of nonzero entries in the main Diagnol

In 3×3 Matricies:

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ In 3×3 matricies

$\det(A)=$

$a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}$

$-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$

Let $A=[a_{ij}]$ be an n×n matrix. The Determinant is defined as follows: $\det(A)=\sum \sigma(p_1,p_2,\ldots,p_n)a_{1p_1}a_{2p_2}$

Some examples of finding the determinant

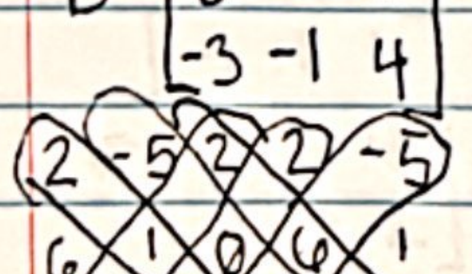
$A=\begin{bmatrix} -4 & 6 \\ -2 & 5 \end{bmatrix}$ $\det(A)=(-4)(5)-(6)(-2)=-8$

$B=\begin{bmatrix} 2 & -5 & 2 \\ 6 & 1 & 0 \\ -3 & -1 & 4 \end{bmatrix}$ $\det(B)=(2)(1)(4)+(-5)(0)(-3)+(2)(6)(-1)-(-3)(1)(2)-$

$(-1)(0)(2)-(4)(6)(-5)$

$=8+0+(-12)-(-6)-0-(-120)$

$=122$



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MATH 165 Midterm #1 Topic List
1.1 Intro to Differential Equations
1.3 Geometry of First Order Differential Equations
1.4 Seperable Differential Equations
1.6 First-Order Linear Differential Equations
1.7 Modeling using Differential Equations
2.1 Intro to Matricies
2.2 Matrix Algebra
2.3 Systems of Linear Equations
2.4 Elementary Row Operations & REF
2.5 Gaussian Elimination
2.6 Inverse of Square Matricies

Useful Formulas:

Seperable Differential Equations

$$\int P(y)\,dy = \int g(x)\,dx + C$$

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3.3: Cofactor Expansion

↳ The general Idea of cofactor expansion is finding the determinenant of large matrix by taking the sum of several smaller matricies

Let A be an $n \times n$ matrix. The cofactor, C_{ij} of the element a_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij}$ Where M_{ij} is the minor of a_{ji}

↳ the cofactor of the main diagnal are positive, so are a checker pattern off of them

→ The rest form a negative checker

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & \vdots & * \end{vmatrix}$$

Cofactor Expansion Theorem:

1. expanding along row i:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{K=1}^{n} a_{ik}C_{ik}$$

2. expanding along column j:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{K=1}^{n} a_{kj}C_{kj}$$

↳ These are called cofactor expansions

Example:

$$\det(A) = \begin{vmatrix} -8 & 7 & 0 \\ 1 & -3 & 6 \\ 4 & -2 & -2 \end{vmatrix}$$

Expanding along row #1

$$-8\begin{vmatrix} -3 & 6 \\ -2 & -2 \end{vmatrix} - 7\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & -3 \\ 4 & -2 \end{vmatrix} =$$

$$= -8(18) - 7(-22) = 10$$

Expanding along column #2:

$$-7\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix} + (-3)\begin{vmatrix} -8 & 0 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} -8 & 0 \\ -1 & 6 \end{vmatrix} = -7(-22) - 3(16) + 2(-48)$$

$$= 10$$

Eigenvalues by Cofactor expansion:

↳ Find the values that satisfy $\det(A - \lambda I_n) = 0$ for

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 3 & 4 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 4 \\ 2 & 0 & 2-\lambda \end{bmatrix}$$

Thus $0 = (-\lambda + 4)(\lambda - 1)(\lambda + 1)$

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The Adjoint Method for finding A^{-1}

↳ If the elements of row i are multiplied by the cofactors of a different row then the sum of the resulting products is zero

1. If we use the elements of row i & the cofactors of row j

$$\sum_{k=1}^{n} a_{ik} C_{jk} = 0 \quad i \neq j$$

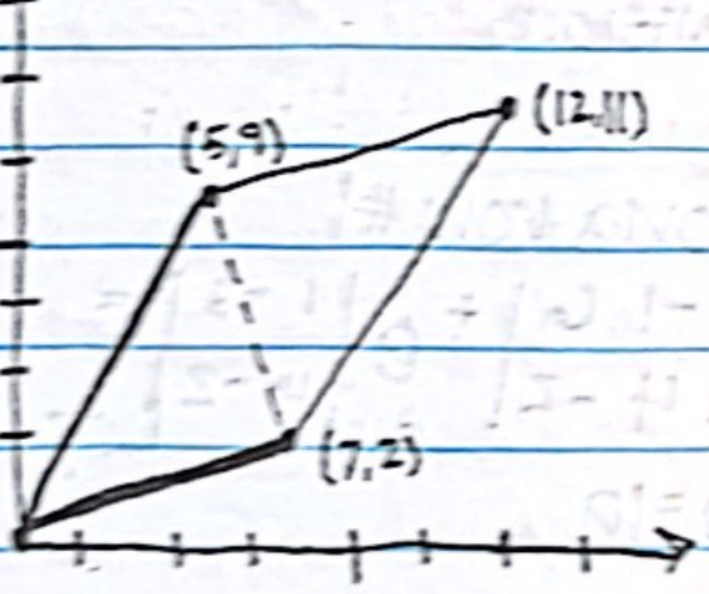
or

$$\sum_{k=1}^{n} a_{ki} C_{kj} = 0 \quad i \neq j$$

Finding the Area of a Parallelogram using Determinants

Consider the Parallelogram with vertices

(0,0) (7,2) (5,9) (12,11)



→ Area determined by

$\begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ → as a matrix P

$\begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix} = P$ $|\det(P)| = \text{Area}$

$10 - 63 = -53$

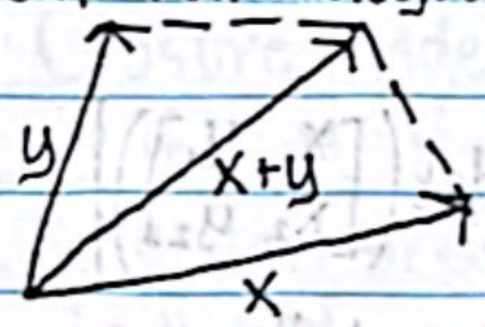
$\text{Area} = 53$

→ Note: 1/2 of the determinant defines a triangle

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4.1 Vector Spaces: Vectors in $\mathbb{R}^n$

Recall: Parallelogram Law

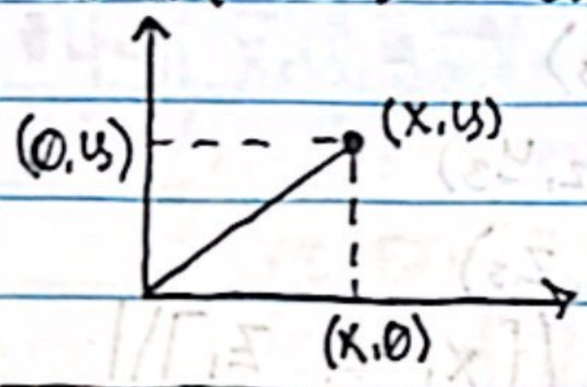


$x+y = y+x$

and

$x+(y+z) = (x+y)+z$

In $\mathbb{R}^2$ Vectors can be identified by an ordered



We know $\mathbb{R}^2$ is the xy-plane & $\mathbb{R}^3$ is 3D

↳ What about $\mathbb{R}^n$?

The set of all vectors w/ real components

$$\mathbb{R}^n = \{\vec{x} = (x_1, x_2, \dots, x_n) \text{ s.t. } x_i \in \mathbb{R}\}$$

Recall: The magnitude of a vector $\vec{x}$ is given by:

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_i^2}$$

Key Properties of vectors in $\mathbb{R}^n$

1. $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$ If $\vec{x} = (x_1, x_2)$ $\vec{y} = (y_1, y_2)$, then $\vec{x}+\vec{y} = (x_1+y_1, x_2+y_2)$
2. If $c \in \mathbb{R}$ is a scalar, then $c\vec{x} = (cx_1, cx_2)$
3. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, Then
   - ↳ $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
   - ↳ $(\vec{x}+\vec{y})+\vec{z} = \vec{z}+(\vec{x}+\vec{y})$
   - ↳ $\vec{x}+\vec{y} \in \mathbb{R}^n$
4. If $\vec{x} \in \mathbb{R}^n$, then there exists $\vec{y} \in \mathbb{R}$ such that $\vec{x}+\vec{y} = 0$

Properties of Scalar Mutiplication: $(s,t \in \mathbb{R})$:

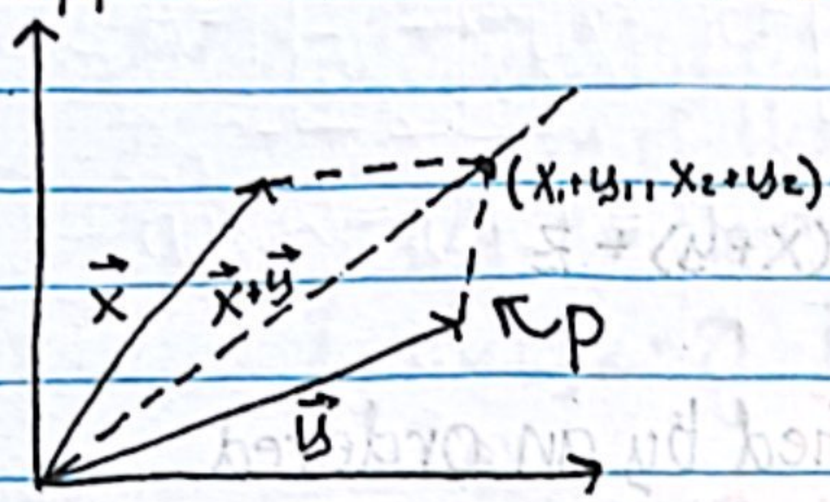
- $1\vec{x} = \vec{x}$
- $(st)\vec{x} = s(t\vec{x})$
- $(s+t)\vec{x} = s\vec{x} + t\vec{x}$
- $s(\vec{x}+\vec{y}) = s\vec{x} + s\vec{y}$

Remember how the determinant can be used to find the area of a parallelogram?

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Geometric Interpretation of Determinants in $\mathbb{R}^n$

Suppose $\vec{x}=(x_1, x_2)$ and $\vec{y}=(y_1, y_2)$ are vectors in $\mathbb{R}^2$



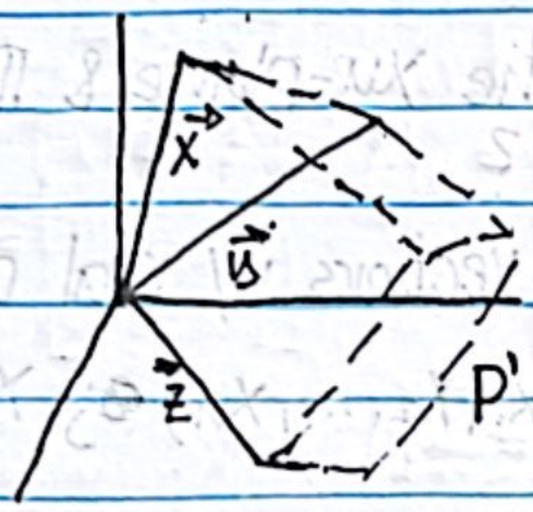
$$\text{Area}(P) = \left|\det\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right)\right|$$

We can extend the same concept to $\mathbb{R}^3$ to find Volume

$\vec{x}=(x_1, x_2, x_3)$

$\vec{y}=(y_1, y_2, y_3)$

$\vec{z}=(z_1, z_2, z_3)$



$$\text{Volume}(P') = \left|\det\left(\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}\right)\right|$$

4.2: General Vector Spaces:

Vector Space (Vek-tor Spā-se) • noun:

Let V be a non-empty set whose elements we will call vectors or functions on those vectors

↳ The set of scalars is called a scalar field

to provide a rigorous definition of a vector space we can look to the 10 Axioms of Vector spaces

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10 Axioms of Vector Spaces:
We can call V a vector space over F provided that it:

★ #1: Closure under vector addition
↳ If $\vec{x}, \vec{y} \in V$, then $\vec{x}+\vec{y} \in V$

★ #2: Closure under scalar multiplication
↳ If $k \in F$ and $\vec{x} \in V$, then $k\vec{x} \in V$

#3: If $\vec{x}, \vec{y} \in V$, then $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

#4: If $\vec{x}, \vec{y}, \vec{z} \in V$, then $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

★ #5: Zero Vector:
there exists $\vec{v} \in V$ such that for all $\vec{x} \in V$, $\vec{x}+\vec{v} = \vec{x}$
↳ denoted $\vec{0}$

#6. Addative Inverses
↳ If $\vec{v} \in V$, then there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v}) = \vec{0}$

#7: Unit Property: $1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$

#8: If $s, t \in F$ then $(st)\vec{v} = s(t\vec{v}) = t(s\vec{v})$

#9: If $s, t \in F$ & $\vec{x}, \vec{y} \in V$, then $s(\vec{x}+\vec{y}) = s\vec{x}+s\vec{y}$

#10: If $s, t \in F$ & $\vec{x} \in V$, then
a. $(s+t)\vec{x} = s\vec{x}+t\vec{x}$

Adjoint Method of finding A^{-1}:
↳ Suppose A is a $n \times n$ matrix and $\det(A) \neq 0$
↳ Let M_c be the $n \times n$ matrix that we get by replacing A_{ij} with its corresponding Cofactor → $(M_c)_{ij} = C_{ij}$
↳ M_c is the adjoint matrix for A ($adj(A)$)
↳ $(adj(A))_{ij} = C_{ji}$

Then $A^{-1} = \frac{1}{\det(A)} adj(A)$

Recall if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ &
$\det(A) = ad - cb \neq 0$
then $A^{-1} = 1/\det(A)$

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Adjoint Inverse Example:

Let $A = \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & -3 \end{bmatrix}$ $M_C = \begin{bmatrix} 6 & 9 & 6 \\ -3 & 18 & -3 \\ -1 & -6 & -10 \end{bmatrix}$

$adj(A) = (M_C)^T = \begin{bmatrix} 6 & -3 & -1 \\ 9 & -18 & -6 \\ 6 & -3 & -10 \end{bmatrix}$ $\det(A) = 27$

↳ Thus $A^{-1} = \frac{1}{27}\begin{bmatrix} 6 & -3 & -1 \\ 9 & -18 & -6 \\ 6 & -3 & -10 \end{bmatrix}$

Cofactor Expansion Example:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $M_{11} = \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{bmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (5)(9)-(6)(8)$ $= -3$

$M_{23} = \begin{bmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{9} \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = (1)(8)-(2)(7)$ $= -6$

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (5)(9)-(8)(6)$ $= -3$

$M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = (4)(9)-(7)(6)$ $= -6$

$M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (4)(8)-(5)(7)$ $= -3$

$\det(A) = (1)(1)M_{11} + (2)(-1)M_{12} + (3)(1)M_{13}$

$= -3 + 12 - 9 = 0$

$\rightarrow \det(A) = 0$ A is not invertable

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Cramer's Rule for Solving Linear Systems:

↳ Suppose we have an $n \times n$ linear system given by $A\vec{x} = \vec{b}$ where $\det(A) \neq 0$

↳ Let B_k be the matrix that we get by replacing the k-th column of A with $\vec{b}$

$$B_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\,k-1} & b_1 & a_{1\,k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,k-1} & b_2 & a_{2\,k+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,k-1} & b_n & a_{n\,k+1} & \cdots & a_{nn} \end{bmatrix}$$

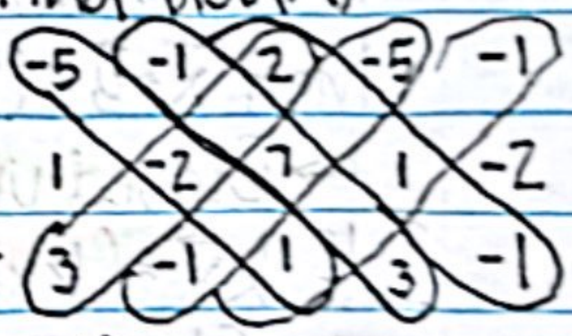
Cramer's Rule:

The solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is given by $x_k = \dfrac{\det(B_k)}{\det(A)}$

Example:

$$\left.\begin{aligned} -5x_1 - x_2 + 2x_3 &= 9 \\ x_1 - 2x_2 + 7x_3 &= -2 \\ 3x_1 - x_2 + x_3 &= -6 \end{aligned}\right\} \quad A = \begin{bmatrix} -5 & -1 & 2 \\ 1 & -2 & 7 \\ 3 & -1 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 9 \\ -2 \\ -6 \end{bmatrix}$$

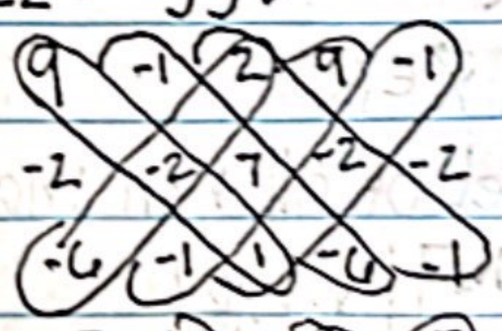
Find $\det(A)$



$\det(A) = \big((-5)(-2) + (-7)(3) + (-2)\big) - \big((3)(-2)(2) + (-5)(-7) + (-1)\big)$

$= -9 - 13 - 22 = -35$ ✓

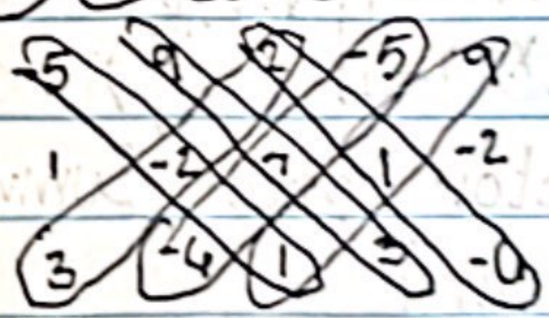
$$\det(B_1) = \begin{vmatrix} 9 & -1 & 2 \\ -2 & -2 & 7 \\ -6 & -1 & 1 \end{vmatrix} =$$



$\big((-18) + (-42) + (4)\big) - \big((24) + (-63) + (4)\big)$

$= 65$

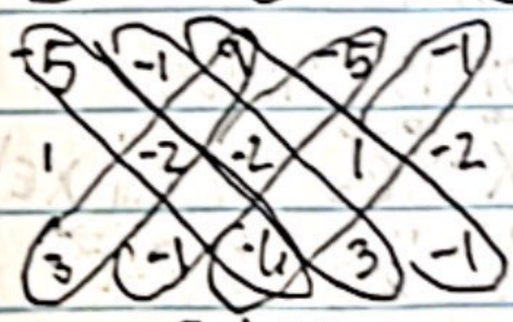
$$\det(B_2) = \begin{vmatrix} -5 & 9 & 2 \\ 1 & -2 & 7 \\ 3 & -6 & 1 \end{vmatrix} =$$



$\big((10) + 7(27) - 12\big) - \big(-12 + 7(30) + 9\big)$

$= -20$

$$\det(B_3) = \begin{vmatrix} -5 & -1 & 9 \\ 1 & -2 & -2 \\ 3 & -1 & -6 \end{vmatrix} =$$



$(-60 + 6 - 9) - (-54 - 18 + 4)$

$= -5$

Thus by Cramer's Rule

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{65}{-35}$$

$$\vec{x} = \begin{bmatrix} \frac{\det(B_1)}{\det(A)} \\ \\ \\ \end{bmatrix}$$

$$x_2 = \frac{-20}{-35} \qquad x_3 = \frac{-5}{-35}$$

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Vector Spaces Continued...

↳ Vector Spaces are all about sums & scalar multiples of vectors $\vec{v} \in V$

Let $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n} \in V$ & $c_1, c_2, \dots, c_n \in F$ then, a <u>Linear Combination</u> of $\vec{v_1}, \dots, \vec{v_n}$ is any vector of the form $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$

Example: Let $A\vec{x} = 0$ be a linear system w/ solution set

$S = \left\{ s\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \mid s,t \in \mathbb{R} \right\}$ → S consists of all linear combos of $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$ & $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$

More Examples of Vector Spaces:

1. $\mathbb{R}^n = \{(x_1 \dots x_i) \mid x_i \in \mathbb{R}\}$
   $\vec{x} = (x_1, \dots, x_n)$ & $\vec{y} = (y_1, \dots y_n)$
   Then $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
   ↳ Closed under vector addition
2. If $k\vec{x} = (kx_1, kx_2, \dots, kx_n) \in \mathbb{R}^n$
   ↳ Closed under scalar multiplication.
   ↳ $\vec{0} = (0, \dots, 0) \in \mathbb{R}^n$
   $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$
3. $V = M_n(\mathbb{R})$, the set of all $n \times n$ matricies w/ real entries
   ↳ $M_2(\mathbb{R}) = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \mid x_i \in \mathbb{R} \right\}$

Important VS:
$\mathbb{R}^n, C^n, M_n(\mathbb{R}), M_n(\mathbb{R})$
$Fn(\mathbb{R}), Ck(I), F(I)$

Properties of Vector Spaces (Summarized)

Let V be a VS over F

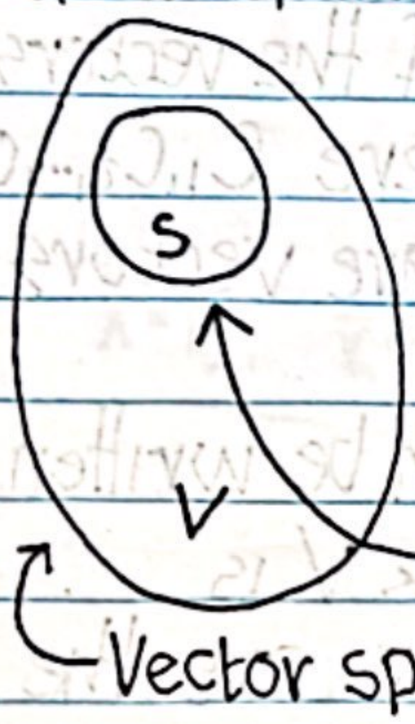
1. The zero vector is unique
   → if $\vec{x} + \vec{0}_1 = \vec{x} - \vec{x}\vec{0}_2$ for all $x \in V$, then $\vec{0}_1 = \vec{0}_2$
2. If $0 \in F$ on $\vec{v} \in V$ then $0\vec{v} = \vec{0}$
3. $k\vec{0} = \vec{0}$ for all $k \in F$
4. Addative Inverses are unique

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4.3: Subspaces:

Let S be a non-empty subset of a vector space V under the same opperations of addition & scalar multiplication as V
If S is also a vector space then S is a subspace of V
↳ Ex: $V=\mathbb{R}^n$, $S=\{x\in\mathbb{R}^n \mid x=(0,x_2,\dots,x_n), x_n\in\mathbb{R}\}$

Take the VS defined by $V=M_n(\mathbb{R})$ &
→ S is a non-empty subset of V, but S is not a vector space so it is Not a subspace → why? $0_n\in S$ $\det(0_n)=0$
→ meaning S does not have a zero-vector and as such is not! a vector space!



⇒ If S is a subspace of V it contains ~ The zero vector! ~
Solution set (If S is indeed a vector subspace)
Vector space of unknowns

The Nullspace of an m×n matrix A is defined as an solution to the homogenous system of the form $A\vec{x}=0$
$\text{nullspace}(A)=\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=0\}$
note: does not hold for nonhomogenous systems!

Let V denote the vector space of all real functions that are defined on an interval I. & let $C^k(I)$ be the set of all functions that are continious and have at least k continious derivatives on interval I. for any fixed interger k (Positive) show that $C^k(I)$ is a subspace → $C^k(I)=\{f\in f, f', f'', \dots, f^{(k)} \text{ exist \& cont on } I\}$
↳ Zero Vector exists!
$0(x)=0$ for all $x\in I$ → Shes a subspace!
NOT COMPREHENSIVE WORK FOR THIS EXAMPLE!

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The set of all solutions to the homogenous linear DE
$y'' + a_1(x)y' + a_2(x)y = \emptyset$ on an interval I
is a vector space!

Often called the **Solution Set**
↳ Every vector in the solution set to the DE
$y'' + a_1(x)y' + a_2(x)y = \emptyset$ can be written as a linear combination of any two nonproportional solutions y_1 & y_2
↳ We say that the solution set is spanned by y_1 & y_2

4.4: Spanning Sets:

Linear Combination: any combination of the vectors in V of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ where $c_1, c_2, \ldots, c_k$ are constants (Scalar) → Linear combos. are vectors

If every vector in a vector space V can be written as a linear combination of $V_1, V_2, \ldots, V_k$ V is **Spanned by** $\underbrace{V_1, V_2, \ldots, V_k}$ & these vectors are the **Spanning Set** of V

Example: Show that $\mathbb{R}^2$ is spanned by the vectors
$\vec{v}_1 = (1,1)$ & $\vec{v}_2 = (2,-1)$
to do this we merely have to establish that for every $\vec{v} = (x_1, x_2)$ there exists constants c_1 & c_2 such that
$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 \rightarrow (x_1, x_2) = c_1(1,1) + c_2(2,-1)$
as a linear system

$$c_1 + 2c_2 = x_1$$
$$c_1 - c_2 = x_2 \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = A$$

$$\det(A) = (1)(-1) - (2)(1) = -3$$

Since $\det(A) \neq 0$ $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ can be satisfied by any $\vec{v} \in \mathbb{R}$ → for all $(x_1, y_1) \in$ the system has a unique solution!

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Spanning Sets Example #2: Verify that the following Matricies span $M_2(\mathbb{R})$

$A_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $A_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$ $A_3=\begin{bmatrix}1&0\\1&0\end{bmatrix}$ $A_4=\begin{bmatrix}1&1\\1&1\end{bmatrix}$

↳ Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ Then, $c_1A_1+c_2A_2+c_3A_3+c_4A_4=A$

↑ This can be written as a system!

$c_1+c_2+c_3+c_4=a$ ←

$c_2+c_3+c_4=b$ Solving by back sub.

$c_3+c_4=c$

$c_4=d$

$c_1=a-b$
$c_2=b-c$
$c_3=c-d$
$c_4=d$

} yields:

$A=(a-b)A_1+(b-c)A_2+(c-d)A_3+dA_4$

Since this equation can be written as a linear combination of the given vectors (A_1, A_2, A_3 & A_4) they do span $M_2(\mathbb{R})$!

↳ The most natural spanning set of $M_2(\mathbb{R})$ is:

$$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

Linear Span: Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V; forming all possible linear combinations of $v_1, v_2, \ldots, v_k$ generates a subset of V called its Linear Span denoted as $\text{span}(v_1, v_2, \ldots, v_k)=\{v\in V: v=c_1v_1+c_2v_2+\cdots+c_kv_k, F\}$

↑ where F is the set of all constants (scalars)

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Linear Span Example: Let $V=C^2(I)$
& $y_1=\sin x$, $y_2=\cos x$ Then,

$$\text{span}(y_1,y_2)=\{y\in C^2(I): y(x)=c_1\cos x+c_2\sin x, \text{ for } c_1,c_2\in\mathbb{R}\}$$

↳ Note: any linear span is a subspace

4.5 Linear Dependence & Linear Independence:

Vector Spaces often have more than 1 spanning set

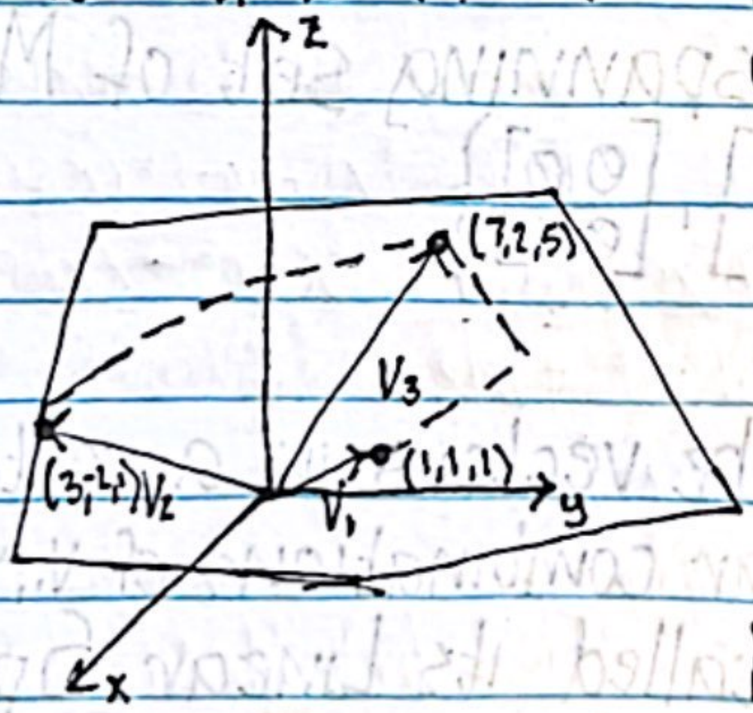
$\{(1,0),(0,1)\}$
$\{(1,0),(1,1)\}$ → All of these are spanning subsets of $\mathbb{R}^2$
$\{(1,0),(0,1),(1,2)\}$

Minimal Spanning Sets are ones with the least number of vectors needed to span the entire space

Consider:

$V_1=(1,1,1)$
$V_2=(3,-2,1)$ → $\det([V_1,V_2,V_3])=0$
$V_3=4V_1+V_2=(7,2,5)$

meaning the vectors lie in a plane & since they aren't colinear → the linear span of these vectors is that of the entire plane they lie in



However...
V_1 & V_2 generate the same plane meaning V_3 is not neccessary & since $\mathbb{R}^3$ cannot be spanned by a single Vector, $\{V_1,V_2\}$ is a minimal spanning set of $\mathbb{R}^3$

Thus, if one of the vectors in a set is a linear combination of other elements in the set then it need not be included in the spanning set!

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A finite nonempty set of vectors $\{V_1, V_2 \dots, V_k\}$ in a vector space V is said to be **linearly dependent** if there exists scalars $c_1, c_2, \dots, c_k$ (not all zero) such that $c_1V_1 + c_2V_2 + \cdots + c_kV_k = 0$

This is referred to as a **Linear Dependence**

If the only value of the scalars that satisfies $c_1V_1 + c_2V_2 + \cdots + c_kV_k = 0$ is $c_1 = c_2 = \cdots = c_k = 0$ the vectors are **linearly Independent**

→ Consider the simple case of a set containing a single vector
If $V = 0$, then $\{V\}$ is linearly dependent for any c_1

$c_1 0 = 0$

However if $V \neq 0$ then $\{V\}$ is linearly Independent since 0 is the only value that a scalar could be to satisfy

$c_1 V = 0$

A set of at least 2 vectors is said to be linearly dependent if & only if at least one of the vectors can be expressed as a linear combination of the other vectors

↳ Minimal Spanning Sets are always linearly Independent & Linearly dependent sets cannot be minimally spanning

Generally, for simple examples, a set of vectors is linearly dependent if

1. The vectors are proportional
   and/or
2. The set contains the zero vector

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Example: Determine if $\{v_1, v_2, v_3\}$ is linearly dependent

$v_1 = (3,-1,2)$ Recall: $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$v_2 = (-9,5,-2)$ $c_1(3,-1,2) + c_2(-9,5,-2) + c_3(-9,9,6) = (0,0,0)$

$v_3 = (-9,9,6)$ which can be represented as a system

$\hookrightarrow 3c_1 - 9c_2 - 9c_3 = 0$

$-c_1 + 5c_2 + 9c_3 = 0 \rightarrow \left[\begin{array}{ccc|c} 3 & -9 & -9 & 0 \\ -1 & 5 & 9 & 0 \\ 2 & -2 & 6 & 0 \end{array}\right]$

$2c_1 - 2c_2 + 6c_3 = 0$

IN RREF

$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Since the bottom row is free that means there are infinitely many solutions to the system making the vectors linearly dependent

Their dependency is defined by $-6v_1 - 3v_2 + v_3 = 0$

Example #2: Determine whether the following set of polynomials is linearly dependent

$p_1(x) = 1 - 4x^3$ $c_1(1-4x^3) + c_2(2+2x) + c_3(1-x^2+2x^3) + c_4(2x-x^3) = 0$

$p_2(x) = 2 + 2x$ As a Matrix

$p_3(x) = 1 - x^2 + 2x^3$ Putting this matrix into RREF yields

$p_4(x) = 2x - x^3$

$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -4 & 0 & 2 & 1 & 0 \end{array}\right]$

the Identity matrix meaning the only solution is when all constants are zero

Linearly Independent

Note: Every set of linearly dependent vectors has a smaller linearly Independent

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Linear Independence of Functions:

↳ Similar to differential equations, we can determine linear dependence on an Interval

The Wronskian: let $f_1, f_2, \dots, f_k$ be functions in $C^{k-1}(I)$, the order k determinant is given by:

↳ Recall: Cofactor Expansion

$$W[f_1, f_2, \dots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

If $W[f_1, f_2, \dots, f_k]$ is nonzero at some point x_0 in I then, $\{f_1, f_2, \dots, f_k\}$ is linearly Independent on I

↳ NOTE! if $W[f_1, f_2, \dots, f_k](x) = 0$ for all x in I then the Wronskian gives no information about the linear dependence of $\{f_1, f_2, \dots, f_k\}$

Example: determine whether the following functions are linearly dependent on $I = (-\infty, \infty)$

$f_1(x) = e^x$, $f_2(x) = x^2 e^x$ } $W[f_1, f_2](x) = \begin{vmatrix} e^x & x^2 e^x \\ e^x & 2xe^x + x^2 e^x \end{vmatrix}$ the only point where this determinat = 0 is when $x = 0$

↳ Linearly Independent! ←

The vectors $v_1, v_2, \dots, v_k$ are linearly Independent if the vector equation $A\vec{x} = \vec{0}$ has exactly one solution at $\vec{x} = \vec{0}$

For any set of vectors in $\mathbb{R}^n$ $\{v_1, v_2, \dots, v_k\}$ if $k > n$, then Rank(A) < n & there is at least 1 free variable & $\{v_1, v_2, \dots, v_k\}$ are linearly Independent → Infinitely many solutions:

If $k = n$, then A is an $n \times n$ matrix & $\{v_1, v_2, \dots, v_k\}$ are linearly Independent if & only if $\det(A) \neq 0$

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4.6: Bases & Dimension:

A set of vectors $\{v_1, v_2, \dots, v_k\}$ in a vector space V is called a **basis** for V if
- a. The vectors are linearly Independent
- b. The vectors span V

Finite Dimensional Vector Space: one that has finite elements in its spanning set (as opposed to **Infinite dimensional**)

Standard Basis: For $\mathbb{R}^n$ let $e_i \in \mathbb{R}^n$ with a 1 in the ith component → $\{e_1, \dots, e_n\}$ is the Standard Basis

→ The SB of $\mathbb{R}^3$ is given by: $\{e_1, e_2, e_3\}$ where

$e_1 = [1,0,0]$ → Spanning set: $(x_1, x_2, x_3) = x_1e_1 + x_2e_2 + x_3e_3$

$e_2 = [0,1,0]$ → Obv. they are linearly Independent

$e_3 = [0,0,1]$

→ The SB of $V = M_2(\mathbb{R})$ is given by the set:

$$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

Spanning Set: $\begin{bmatrix}c_1&c_2\\c_3&c_4\end{bmatrix} = c_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + c_2\begin{bmatrix}0&1\\0&0\end{bmatrix} + c_3\begin{bmatrix}0&0\\1&0\end{bmatrix} + c_4\begin{bmatrix}0&0\\0&1\end{bmatrix}$

Linearly Independent?

$\begin{bmatrix}0&0\\0&0\end{bmatrix} = \begin{bmatrix}c_1&c_2\\c_3&c_4\end{bmatrix}$ → only 0 when $c_1 = c_2 = c_3 = c_4 = 0$

Dimension (denoted $\dim(V)$) is the number of vectors in any basis of a finitely dimensional vector space V

↳ note: if $V = \{\vec{0}\}$ then $\dim(V) = 0$

↳ All basises in a given V have the same dimension (assuming that it is not Infinite)

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If $\dim(V)=n$. then any n linearly Independent vectors $\subset V$ is a valid basis for V, & any set of vectors that span V is linearly Independent

If $\dim(V)=n$, any set of more than n vectors is neccesarily Linearly Dependent

Example: the standard basis of $P_n(\mathbb{R})$ is given by:
$\{1, x, x^2, \ldots, x^n\}$ → which is infinitely dimensional

~~Important dimensions~~
Important Dimensions to remember
$\dim[\mathbb{R}^n]=n$
$\dim[M_{m\times n}(\mathbb{R})]=mn$
$\dim[M_n(\mathbb{R})]=n^2$
$\dim[P_n(\mathbb{R})]=n+1$

In Summary: if $\dim(V)=n$ & $S=\{v_1,\ldots,v_n\}$ is a set of n vectors then the following statements are equivaelent: S is a basis for V
S is L.Independent
S span> V

Suppose that V is a VS with $\dim(V)=n$ & $S=\{v_1,\ldots,v_k\}$:

S	$k<n$	$k=n$	$k>n$
linearly Independent?	maybe	maybe	No
Spans V?	No	maybe	Maybe
Is a basis?	No	maybe	No

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Example: Let S denote the subspace of $M_2(\mathbb{R})$ consisting of all symetric 2x2 matricies determine a basis for S & find dim(S). Extend this basis to get a basis for $M_2(\mathbb{R})$

↳ In set notation $S=\{A\in M_2(\mathbb{R}): A^T=A\}$

Which can be rewritten in terms of the elements of A

$$S=\left\{\begin{bmatrix}a&b\\b&c\end{bmatrix}: a,b,c\in\mathbb{R}\right\} \text{ \& } \begin{bmatrix}a&b\\b&c\end{bmatrix}=a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Thus, $S=\text{span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ → These matricies are obv. linear Independent

Thus, this set is a basis for S ↑

& Since there are 3 vectors in the basis dim(S)=3

Recall $\dim[M_2(\mathbb{R})]=4$, so we need to add one more vector to the set such that it spans $M_2(\mathbb{R})$ without creating any linear dependencys → $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}$ ← This matrix is the simplest that completes the basis but there are several correct answers

Let V be vector space such that dim(V)=n & Let S be $\{v_1, v_2, \ldots, v_n\}$ be a collection of vectors in V. then the following are equivalent

1. S is linearly Independent
2. S is a basis for V
3. S is a spanning set for V

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MATH 165 HW # 7.

1. Prove that $W=\{p(x)\in P_1(\mathbb{R}) \mid 2p'(0)=p(1)\}$ is a sub space of $P_1(\mathbb{R})$. determine a basis for W, determine dim(W)

$p(x)=?\ x+1 \rightarrow p(1)=2$
$p'(x)=?\ 1 \rightarrow 2p'(0)=2$
$\rightarrow 2p'(0)=p(1)$

↑ we can use this as a general equation to prove that W is a subspace of $P_1(\mathbb{R})$

Closure?

$a(x+1)=a(p(x))$
$=ax+a \rightarrow \frac{d}{dx}=a \rightarrow (2p'(0))a=(p(1))a$ multiplication ✓
↳ $2a=a+a$ ↵

$(x+1)+b$ $2p'(0)+b=p(1)+b$
$\frac{d}{dx}=1$ ↳ $2+b=2+b$ ↵ addition ✓

Zero Vector?

$x+1+0(p(x))=x+1$ ✓ ⟶ She's a subspace!

Basis of W:

- ☑ Linearly Independent $C_1(x+1)=\{(x+1)\}=Basis(W)$
- ☑ Spans $P_1(\mathbb{R})$ $C_1(x+1)=0$
  ↳ can make any $C_1=$ scalar
  $p(x)$ st $2p'(0)=p(1)$ W/ Scalar ∈ V

$\dim(W)=1$

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2. Determine a basis for the following subspaces

$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 2 & -4 & -9 & 5 \end{bmatrix}$

b. $V = \mathbb{R}^4$, $W = Null(A)$

$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 0 & -6 & -9 & 3 \end{bmatrix}$

a) $V = M_3(\mathbb{R})$, $W = \{A \in M_3(\mathbb{R}) \mid A^T = -A\}$

$\begin{bmatrix} a & x & d \\ b & y & e \\ c & z & f \end{bmatrix} = A \Rightarrow A^T = \begin{bmatrix} a & b & c \\ x & y & z \\ d & e & f \end{bmatrix}$

$\begin{bmatrix} a & b & c \\ x & y & z \\ d & e & f \end{bmatrix} = \begin{bmatrix} -a & -x & -d \\ -b & -y & -e \\ -c & -z & -f \end{bmatrix}$

$a = -a \rightarrow 0$

$-x = b \rightarrow$ 1 & -1

$c = -d \rightarrow$ 1 x -1

$y = -y \geq 0$

$f = -f \rightarrow 0$

$z = -z$

$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$

$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$

= basis W

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3. determine whether the set $S=\{(1,2,0,-2),(1,0,3,2),(0,-2,1,1)\}$ is linearly independent & if $(1,1,1,1)$ is in its span.

$c_1(1,2,0,-2)+c_2(1,0,3,2)+c_3(0,-2,1,1)=0$

$$\begin{bmatrix}1&1&0\\2&0&-2\\0&3&1\\-2&2&1\end{bmatrix}=0$$ $(1,1,1,1)$ is NOT in span S

$$\left[\begin{array}{ccc}1&1&0\\2&0&-2\\0&3&1\\-2&2&1\end{array}\right]\rightarrow\left[\begin{array}{ccc}1&1&0\\2&0&-2\\1&0&0\\-2&2&1\end{array}\right]\rightarrow\left[\begin{array}{ccc}1&1&0\\0&0&1\\1&0&0\\0&2&-1\end{array}\right]\rightarrow\left[\begin{array}{ccc}1&1&0\\0&0&1\\1&0&0\\0&2&-1\end{array}\right]\rightarrow\left[\begin{array}{ccc}1&-1&1\\0&0&1\\1&0&0\\0&2&-1\end{array}\right]\rightarrow\left[\begin{array}{ccc}1&-1&0\\0&0&1\\1&0&0\\0&2&-1\end{array}\right]$$

$$RREF=\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{array}\right]$$ → Linearly Independent!

4a Suppose Rank(A)=5. Is S linearly Independent?
Not enough information. the # above the leading terms do not impact rank but could affect the linear dependencies in an unspecified way

4b. Suppose Rank(A)=3
They are linearly independent!
If Rank(A) < n# rows there must be a free variable meaning there are Infinitely many solutions!

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4.8: Row & Column Space:

Let $A=[a_{ij}]$ be a $m \times n$ real matrix. The row vectors of this matrix are row n-vectors in $\mathbb{R}$ and can therefore be associated with vectors in $\mathbb{R}^n$. The sub space of $\mathbb{R}^n$ Spanned by these vectors is called the row space

$A=\begin{bmatrix} 2 & -1 & 3 \\ 5 & 9 & -7 \end{bmatrix}$ then,

$\text{rowspace}(A)=\text{span}\{(2,-1,3),(5,9,-7)\}$

We say two matricies are row equivaelent if

$\text{rowspace}(A)=\text{rowspace}(B)$

4.9: The Rank-Nullity Theorem

Recall: The null space of A is defined to be the set of all real solutions to the homogenous sysusystem.

$\text{nullspace}(A)=\{x \in \mathbb{R}^n : Ax=0\}$

the dimension of nullspace(A) is reffered to as the nullity of A

↳ When try to find the nullity we first need to find the basis of the nullspace

Rank Nullity Theorem: For any $m \times n$ matrix A.

$\text{Rank}(A)+\text{nullity}(A)=n$

→ Example:

Find the matrix's nullity

$A=\begin{bmatrix} 2 & -6 & -8 \\ -1 & 3 & 4 \\ 5 & -15 & -20 \\ -2 & 6 & 8 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 & -4 & 0 \\ -1 & 3 & 4 & 0 \\ 5 & -15 & -20 & 0 \\ -2 & 6 & 8 & 0 \end{bmatrix} \leftarrow \rightarrow \begin{bmatrix} 1 & -3 & -4 & 0 \\ -1 & 3 & 4 & 0 \\ 5 & -15 & -20 & 0 \\ -2 & 6 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, $M_1(1/2), A_{12}(1), A_{13}(-5), A_{14}(2)$

Who's Space is Who's?	Subspace of	Dim
nullspace(A) → set of all vectors x w/ Ax=0	$\mathbb{R}^n$	nullity(A)
rowspace(A) → span of the vectors of A	$\mathbb{R}^n$	rank(A)
colspace(A) → span of the column vectors of A	$\mathbb{R}^m$	rank(A)

Systems of Linear equations:

Let A be a m×n matrix, & consider the corresponding homogenous linear system $Ax=0$

1. If rank(A)=n, then $Ax=0$ has only the trivial solution & the nullspace(A)={0}
2. If rank(A)=r<n, then $Ax=0$ has an Infinite number of solutions all of which can be expressed in the form

$$x = c_1x_1 + c_2x_2 + \cdots + c_{n-r}x_{n-r}$$

Where $\{x_1, x_2, \ldots, x_{n-r}\}$ is any linearly Independent set of n-r solutions to $Ax=0$

Let A be an m×n & consider the linear system $Ax=b$

1. If b is not in colspace(A), then the system is inconsis
2. If $b \in$ colspace(A), then the system is consistent & has the following
   a. a unique solution if & only if dim(colspace(A))=n
   b. an infinite number of solution if & only if dim(colspace(A))<n

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If S is a subspace of V then dim(S) ≤ dim(V)

Basis extension:

Theorem: Let S be a subspace of V, any basis for S is part of a basis of V → Assuming V is finitely dimensional

↳ We can find the basis of V by "extenting" the basis of S by adding vectors in V that are not in the basis of S until we have a linearly independent

Example: Let S be the set of all 2x2 symetric matricies in $M_2(\mathbb{R})$

Find a spanning set

$S = \{M \in M_2(\mathbb{R}) \mid M = \begin{bmatrix} a & b \\ b & d \end{bmatrix}\ a, b, d \in \mathbb{R}\}$

Row Spaces Continued:

↳ The set of non-zero row vectors for any REF of A is a basis is a basis for rowspace(A)

↳ If B is a REF of A then the columns of A that correspond to the columns of B with leading 1s is a basis for the colspace(A)

Example:

$$A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 & 5 \\ -2 & -7 & 5 & -5 & -6 & -9 \\ -2 & -6 & 3 & -6 & -6 & -8 \\ 1 & 5 & 2 & 1 & 3 & 6 \end{bmatrix}$$ Find a basis for rowspace(A) & colspace(A)

$$\rightarrow \begin{bmatrix} 1 & 4 & -1 & 2 & 3 & 5 \\ 0 & 1 & 3 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for colspace(A):

$\{(1,-2,-2,1), (4,-7,-6,5)\}$

Basis for rowspace(A)

$\{(1,4,-1,2,3,5), (0,1,3,-1,0,1)\}$

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MATH 165 Midterm 2 Topic List:

3.1 Intro. to the Determinant
3.2 Properties of the Determinants
3.3 Cofactor Expansion
4.1 Vectors in $\mathbb{R}^n$
4.2 Definition of a vector space
4.3 Subspaces
4.4 Spanning Sets
4.5 Linear Dependence & Linear Independence
4.6 Bases & Dimension
4.7 Change of Basis
4.8 Row & Column Space
4.9: Rank Nullity Theorem

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Spring 2019 Midterm #2:

1. Consider the matrix: find the inverse

$$A=\begin{bmatrix}1&-1&2\\2&-3&3\\1&-1&1\end{bmatrix}\rightarrow\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\2&-3&3&0&1&0\\1&-1&1&0&0&1\end{array}\right]\xrightarrow{R_1-R_3}\left[\begin{array}{ccc|ccc}0&0&1&1&0&-1\\2&-3&3&0&1&0\\1&-1&1&0&0&1\end{array}\right]$$

$$\xrightarrow{R_2-2R_3}\left[\begin{array}{ccc|ccc}0&0&1&1&0&-1\\0&-1&1&0&1&-2\\1&-1&1&0&0&1\end{array}\right]\xrightarrow{R_3-R_2}\left[\begin{array}{ccc|ccc}0&0&1&1&0&-1\\0&0&1&0&1&-2\\1&0&0&0&-1&-1\end{array}\right]$$

$$\xrightarrow{R_2-R_1}\left[\begin{array}{ccc|ccc}0&0&1&1&0&-1\\0&-1&0&-1&1&-3\\1&0&0&0&-1&-1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}0&0&1&1&0&-1\\0&1&0&1&-1&3\\1&0&0&0&-1&-1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}$$

Inverse of A $\rightarrow$

$$\left[\begin{array}{ccc|ccc}0&-1&-1&1&0&0\\1&-1&3&0&1&0\\1&0&-1&0&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}0&-1&3&1&0&-1\\1&-1&1&0&1&-2\\1&0&-1&0&0&1\end{array}\right]$$

Using the Inverse solve the system:

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&0&-1&3\\0&1&0&1&-1&1\\0&0&1&1&0&-1\end{array}\right]\rightarrow x\begin{bmatrix}0\\1\\1\end{bmatrix}+y\begin{bmatrix}-1\\-1\\0\end{bmatrix}+z\begin{bmatrix}3\\1\\-1\end{bmatrix}.$$

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$$B=\begin{bmatrix}2&-1&3&1\\1&4&-2&3\\0&2&-1&0\\1&3&-2&5\end{bmatrix}$$

$(2\cdot4\cdot-1\cdot5)+(-1\cdot-2\cdot0\cdot1)+(3\cdot3\cdot0\cdot3)+(1\cdot1\cdot-2\cdot2)-((1\cdot2\cdot-2\cdot1)+(3\cdot-1\cdot3\cdot2)+(-2\cdot0\cdot1\cdot-1)+(5\cdot0)$

$-40 \quad -24 \quad -4 \quad -10$

$-44+22=-22$

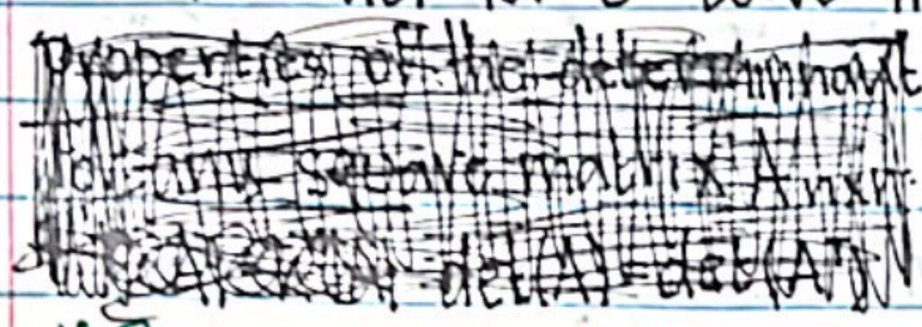
5/10

1. Let A, B and $C \in M_{n\times n}(\mathbb{R})$ be that C is an invertable matrix and $AB=C^3$. Show that B is an invertable matrix

↳ note: invertable means $\det(\blacksquare)\neq 0 \leftarrow \det(C)$

5/10

In order for C^3 to be invertable



If B was not invertable

$\det(B)=0$ & $\det(A)0\neq\det(C^3)$ →

→ By definition of the inverse

If C^3 is an invertable matrix then there must exist a matrix C^{-1} such that $C\cdot C^{-1}=I$

Properties of Determinants:	Properties of ~~Determinants~~ Inverse matricies:
Reflection: $\det(A)=\det(A^T)$	1. A is invertable
Repitition: if all elements of a row/col are identical to another row/col then $\det(A)=0$	2. There exists a matrix B such that $BA=I$
Switching any two rows/cols makes det(A) flip signs	3. There exists a matrix C such that $AC=I$
Triangle: If all entries above or below the main diagonal are zero the diagol	4. The equation $A\vec{x}=\vec{b}$ has exactly one solution for every $n\times1$ vector in b
	5. The equation $A\vec{x}=\vec{0}$ has exactly one solution
	↑ 1-5 are equivalent

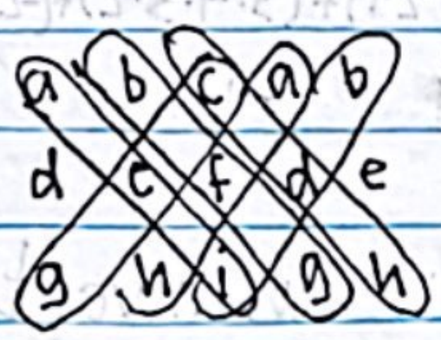
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0/10

2. (10 points) Given that

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 1 \qquad \det\begin{bmatrix} 2a+d & 2b+e & 2c+f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{bmatrix}$$

0/10



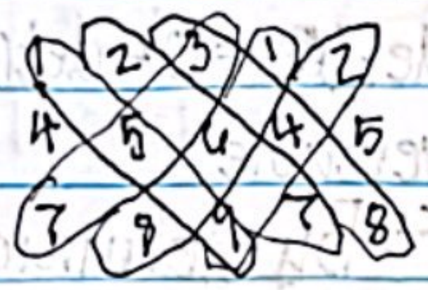
$aei + bfg + cdh - gec - hfa - idb = 1$

↳ this is a linear transformation of A

scalar: 2x

addition: R_1+R_2, R_2+R_3, R_3+R_1

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$



$45+84+96-105-\cancel{48}-72$

$\det(B) = 0$

↳ Keep turnin for full solution!

det of the transformed matrix = ~~2~~ = 9

10/20

3. (20 points) For the following determine if they are valid subspace

a. $V = \mathbb{R}^4$ and $W = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_2 = x_3 + x_4\}$

Zero vector?

→ $0(x_1+x_2) \pm 0(x_3+x_4)$ ✓

Unit vector?

$1(x_1+x_2) = 1(x_3+x_4)$ ✓

Closure?

addition: $x_1+x_2+9 = x_3+x_4+9$ ✓

$\vec{J} = x_1+x_2+x_3+x_4 = x_3+x_4+x_3+x_4$ ✓

multiplication: $a(x_1+x_2) = a(x_3+x_4)$ ✓

$ax_1+ax_2 = ax_3+ax_4$ ✓

yes! W meets the conditions to be a vector subspace of V

10/10

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b. $V=P_4(\mathbb{R})$ & $W=\{p\in V \mid p(1)p(2)=2p(3)\}$

$ax^4+bx^3+cx^2+dx+e=p(x)$ → Yes! W is a vector sub space of V

$p(1)=a+b+c+d+e$

$p(2)=16a+8b+4c+2d+e$

$2p(3)=(81a+27b+9c+3d+e)2=162a+54b+18c+6d+2e$

$(a+b+c+d+e)(16a+8b+4c+2d+e)=162a+54b+18c+6d+2e$

$\hookrightarrow 16a+8b+4c+2d+e=162+54+18+6+2$

$16a+8b+4c+2d+e=242$

$16a+8b+4c+2d+e-242=0$

$16ax^4+8bx^3+4cx^2+2dx+e-242$

Zero Vector?

↳ yes! The solution for x to $16ax^4+8bx^3+4cx^2+2dx+e=242$ will give a vector that when added to any other vector will return the vector it was added to

i.e. if $e=242$ then $x=0$ is the zero vector ✓

Closure?

↳ Addition: $2p(3)+h=p(1)p(2)+h$

$16ax^4+8bx^3+4bx^2+2dx+e+h=242+h$ ✓

Multiplication?: $j(2p(3))=j(p(1)p(2))$

$j(16ax^4+8bx^3+4bx^2+2dx+e)=j(242)$ ✓

Unit vector? $1\cdot(2p(3))=1\cdot p(1)p(2)$ ✓

4. (10 points): Let $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $A_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $A_2=\begin{bmatrix}1&0\\1&2\end{bmatrix}$ & $A_3=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$

determine whether $A\in \text{Span}\{A_1,A_2,A_3\}$ or not

0/10 5.10 Points: $V = M_{3\times2}(\mathbb{R})$

$W = \{A \in V \mid a_{11} + a_{22} = 3a_{12} + 2a_{32}\}$

Find a spanning set for W.

↳ Recall definition of Spanning Set:

the set of linear combinations that can express any vector in the subset

$$\begin{bmatrix} a_{11} & \frac{1}{3}a_{12} \\ \vdots & a_{22} \\ \cdots & \frac{1}{2}a_{32} \\ \vdots & \vdots \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Spanning subset review:

Let $S = \{x^2+1, x-2, 2x^2-x\}$ show that S describes a Spanning set of $\mathbb{P}_2$.

↳ Let $p(x) = ax^2+bx+c$ → this is the most general polynomial with degree ≤ 2

In order to show that S is a spanning set we need to show that some combination of scaled elements of S can represent every polynomial in $\mathbb{P}_2$ → write p(x) (general polynomial in terms of S scaled)

$$p(x) = ax^2+bx+c = r(x^2+1) + s(x-2) + t(2x^2-x)$$

$$= rx^2 + r + sx - 2s + 2tx^2 - tx$$

$$= (r+2t)x^2 + (s-t)x + (r-2s)$$

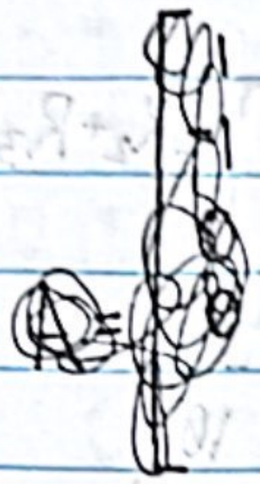
Thus, ←

$$\left.\begin{aligned} a &= r+2t \\ b &= s-t \\ c &= r-s \end{aligned}\right\} \text{ as a matrix:} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \rightarrow \text{magic} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}a+2b+\frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a-\frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a-b-\frac{1}{4}c \end{array}\right]$$

This shows that a solution can exist for any a, b, c Thus, S spans $\mathbb{P}_2$ ←

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Here is a really evil example of column & null space



$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

→ we need to find the dependent columns of A
← ↳ 2 & 6; 4 & 8
this was obvious just from inspection but we need to put it in RREF to identify the pivot

$$RREF(A)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ The first non-zero entry in any row is called a **pivot**
↳ the rows/columns that contain a pivot are called the **pivot rows/columns** of A

→ When a matrix is in RREF the pivot columns are **linearly independent**
↳ Additionally, the columns in the RREF with a single one that follows the main diagonal correspond to the columns in the original matrix that form its basis of the **col space**
↳ In our matrix columns {1, 2, 3, 4, 5, 7} constitute a basis for A ↵

In order to find a basis for the **nullspace** A we need to evaluate A in terms of pivot variables & free variables
↳ there are evidently $n-r$ free variables in A
to solve find $Ax=0$ by <u>expressing each pivot in terms of the free variables</u>

$$\left.\begin{array}{l} X_7=0 \\ X_5=0 \\ X_4=X_8 \\ X_3=0 \\ X_2=X_6 \\ X_1=0 \end{array}\right\}$$ or as a vector →

$$X=X_6\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}+X_8\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(Linear Combination)

Where X_6 & X_8 are free → meaning they can be an $\mathbb{R}$

Notice that null(A) has dimension = 2 and there are 6 pivots which add to $n=8$ ↑
Rank-Nullity Theorem →

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Prob (10 Points) Is the set of polynomials linearly independent

$\{x^3-x,\ 2x^2+4,\ -2x^3+3x^2+2x+6\} \subseteq P_3$

$-2x^3+3x^2+2x+6$ } As a matrix $\longrightarrow$ $\begin{bmatrix} -2 & 3 & 2 & 6 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$ $2R_2+R_3$

$x^3-x+0x+0$

$0+2x^2+0x+4$

$\begin{bmatrix} -2 & 3 & 2 & 6 \\ 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$ R_1+R_2, R_3-R_2 $\longrightarrow$ $\begin{bmatrix} 0 & 3 & 3 & 10 \\ 2 & 0 & 0 & 4 \\ -2 & 2 & 0 & 0 \end{bmatrix}$ R_2-R_1 $\longrightarrow$ $\begin{bmatrix} 0 & 3 & 3 & 10 \\ 2 & -3 & -3 & -6 \\ -2 & 2 & 0 & 0 \end{bmatrix}$

R_2+R_2 $\longrightarrow$ $\begin{bmatrix} 0 & 3 & 3 & 10 \\ 2 & -3 & -3 & -6 \\ 0 & -1 & -3 & -6 \end{bmatrix}$ R_3+R_1 $\longrightarrow$ $\begin{bmatrix} 0 & 3 & 3 & 10 \\ 2 & -3 & -3 & -6 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ R_2+R_3 $\longrightarrow$ $\begin{bmatrix} 0 & 3 & 3 & 10 \\ 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

R_1-3R_3, $\frac{1}{2}R_2 \longrightarrow$ $\begin{bmatrix} 0 & 0 & 3 & 4 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4/3 \end{bmatrix}$ RREF!

$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & 3 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4/3 \end{bmatrix}$

Q $x_1+2x_4=0 \Rightarrow x_1=-2x_4$ } $x_1=x_2$

$x_2+2x_4=0 \Rightarrow x_2=-2x_4$ } $\boxed{.5x_3=x_1=x_2=-2x_4}$

$x_3+4/3x_4=0 \Rightarrow x_3=-4/3x_4 \Rightarrow 3x_3=-4x_2$

not linearly independent

Fall 2019 Midterm#2: Determine whether each given set S is a sub space of the given vector space V. If so, give a proof; if not state a propverty that it fails to satisfy (20 points)

5/5 a. (5 points) $V=\mathbb{R}^2$, & $S=\left\{\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2 \mid x=y-1\right\}$

↳ $y=x+1$

Closure?

addition: $y+c=(x+c)+1$ ✓

multiplication: $a(y)=a(x)+1$

$a(3)=a(2)+1$ X ← Fails scalar closure

0/5 b. (5 Points) $V=\mathbb{R}^3$,

$S=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\in\mathbb{R}^3 \mid x-y=z\right\}$ Closure? Addition?

zero vector?

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $x-y=0=z$ ✓

$0-0=0$ ✓

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\y\\x-y\end{bmatrix}\begin{matrix}x+c\\y+c\\z+c\end{matrix}$

$z+c=(x+c)-(y+c)$

$z+c=x+c-y-c$

Fails closure! → NO! X

5/5 c. (5 Points) $V=M_2(\mathbb{R})$, & $S=\left\{A\in M_2(\mathbb{R}) \mid A^T=-A\right\}$ → Shes a sub-space

$A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $A^T=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ → Generalize $\begin{bmatrix}0&-a\\a&0\end{bmatrix}$

0 vector? ✓

$\begin{bmatrix}0&0\\0&0\end{bmatrix}=A$

closure under addition

$\begin{bmatrix}0&a\\-a&0\end{bmatrix}+\begin{bmatrix}0&b\\-b&0\end{bmatrix}=\begin{bmatrix}0&a+b\\-a-b&0\end{bmatrix}\begin{bmatrix}0&-a-b\\a+b&0\end{bmatrix}$ ✓

Scalar? ✓

Identity? yes!

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5/5 d. (5 points) $V=\mathbb{P}_2$ $S=\{f\in\mathbb{P}_2 \mid f(0)=1\}$

$ax^2+bx+c=f(x)$
$2ax+b=f'(x)$
$b=1=f'(0)$
$\Rightarrow f(x)=2ax+x+c$

Zero vector?
$2a(0)+0+c=0$
↳ yes if $c=0$

NOT A SUB SPACE!

4/5 2. (16 points) Compute the wronkian derivate of the functions

$\{t, 2t+3t^2, t^2-1\} \rightarrow \begin{bmatrix} t & 2t+3t^2 & t^2-1 \\ 1 & 2+6t & 2t \\ 0 & 6 & 2 \end{bmatrix}$ → Linearly Dependent!

$2\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} t^2-1 \\ 2t \\ 2 \end{bmatrix} = \begin{bmatrix} 2t+3t^2 \\ 2+6t \\ 6 \end{bmatrix}$

b. (8 points) let $a_1=\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $a_2=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ & $b=\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$

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Give a non-zero vector x in the nullspace of A

$$A=\begin{bmatrix}-5&-3&4&5&3\\2&-5&4&-4&-4\\-4&1&4&-4&4\end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix}-3&-8&8&1&-1\\2&-5&4&-4&-4\\-4&1&4&-4&4\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}-3&-8&8&1&-1\\5&3&-4&-5&-3\\-4&1&4&-4&4\end{bmatrix}$$

$$\xrightarrow{R_3-R_1} \begin{bmatrix}-3&-8&8&1&-1\\5&3&-4&-5&-3\\-1&-7&-4&-5&-3\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}-3&-8&8&1&-1\\5&3&-4&-5&-3\\3&5&0&0&0\end{bmatrix} \xrightarrow{\substack{R_1+R_3\\R_2+R_2}} \begin{bmatrix}0&-8&8&1&-1\\8&8&-4&5&-3\\3&5&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{bmatrix}0&-8&4&1&-1\\8&0&4&6&-4\\3&5&0&0&0\end{bmatrix} \xrightarrow{\substack{R_2\leftrightarrow R_1\\\frac{1}{2}R_2}} \begin{bmatrix}4&0&2&3&-2\\0&-8&4&1&-1\\3&5&0&0&0\end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix}4&0&2&3&-2\\3&-3&4&1&-1\\3&5&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix}1&3&-2&2&-1\\3&-3&4&1&1\\3&5&0&0&0\end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix}1&3&-2&2&-1\\4&0&2&3&0\\3&5&0&0&0\end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix}-4&-2&-2&2&-1\\4&0&2&3&0\\3&5&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix}2&-2&0&5&1\\4&0&2&3&0\\3&5&0&0&0\end{bmatrix} \begin{bmatrix}2&-2&2&8&1\\4&0&2&3&0\\3&5&0&0&0\end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix}2&-2&2&8&1\\1&-5&2&3&0\\3&5&0&0&0\end{bmatrix}$$

$$A=\begin{bmatrix}-5&3&8&1&3\\2&-5&4&-4&-4\\-6&6&0&0&0\end{bmatrix} \xrightarrow{R_2+R_2} \begin{bmatrix}-5&3&8&1&3\\-4&1&4&-4&-4\\-6&6&0&0&0\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}-1&2&4&5&7\\-4&1&4&-4&-4\\-6&6&0&0&0\end{bmatrix}$$

$$\xrightarrow[+R_3]{R_2-R_3} \begin{bmatrix}-1&2&4&5&7\\2&-5&4&-4&-4\\-1&1&0&0&0\end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix}-1&2&4&5&7\\1&-4&4&-4&-4\\-1&1&0&0&0\end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix}0&-2&8&1&3\\1&-4&4&-4&-4\\-1&1&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix}0&-2&8&1&3\\0&-3&4&-4&-4\\-1&1&0&0&0\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}0&1&4&5&7\\0&-3&4&-4&-4\\-1&1&0&0&0\end{bmatrix} \xrightarrow{R_2+3R_1} \begin{bmatrix}0&1&4&5&7\\0&0&16&11&17\\-1&1&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&-1&0&0&0\\0&1&4&5&7\\0&0&16&11&17\end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix}1&-1&0&0&0\\0&1&-12&-6&-10\\0&0&16&11&17\end{bmatrix}$$

$$\begin{bmatrix}1&0&-12&-6&-10\\0&1&-12&-6&-10\\0&0&16&11&17\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&-12&-6&-10\\0&1&4&5&7\\0&0&16&11&17\end{bmatrix}$$

$$\begin{bmatrix}1&0&4&5&7\\0&1&4&5&7\\0&0&16&11&17\end{bmatrix} \begin{bmatrix}1&0&4&5&7\\-1&1&0&0&0\\0&0&16&11&17\end{bmatrix}$$

$-7\ -7\ 8\ 1\ 3 \qquad -2\ 0\ 8\ 1\ 3$

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1a. (6 Points) Compute the determinant of the following matrix

$$A=\begin{bmatrix} 0 & -1 & 5 & 3 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & -11 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$\det(A) = 0$

b. $-2x_2 + 5x_3 + 3x_4 = 0$
$4x_4 = 0$
$x_1 + x_2 - 11x_3 = 0$
$-3x_3 + x_4 = 0$

$$\begin{bmatrix} 0 & -2 & 5 & 3 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & -11 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \quad R_2 - 7_2 \longrightarrow \begin{bmatrix} 0 & -2 & 5 & -4 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & -11 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix}$$

$R_1 + R_4$

$$\begin{bmatrix} 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & -11 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -11 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -11 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -11 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -11 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -11 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -11 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 - 11x_3 = 0 \quad x_1 = 0$
$x_2 = 0$
$x_2 + x_3 = 0 \rightarrow x_3 = 0$
$x_4 = 0$

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Find a basis for the null space of the matrix

$$A=\begin{bmatrix}1&2&0&-2\\0&-3&4&2\\-1&-5&4&0\end{bmatrix}\quad\begin{bmatrix}1&2&0&-2\\0&-3&4&-2\\0&-3&4&-2\end{bmatrix}\rightarrow\begin{bmatrix}1&2&0&-2\\0&-3&4&-2\\0&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&-1&4&-4\\0&-3&4&-2\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}$$

Determine whether the following are valid subspaces

$V=\mathbb{R}^2$ & $S=\{(x,y)\in V \mid x\leq y\}$

→ Zero vector?

$x=0$ $0\leq 0$ ✓

$y=0$

↳ Shes a subspace!

→ Closure?

addition

$x=a$ $a\leq b$

$y=b$ $c\in\mathbb{R}$

$x+c\leq b+c$ ✓

multiplication:

$ca\leq cb$ ✓

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(16 Points) Determine if $\{\sin(x), \cos(x), e^x\}$ on the interval $(-\pi/2, \pi/2)$ are linearly Independent or not

→ Wronskian: $\begin{bmatrix} \sin(x) & \cos(x) & e^x \\ \cos(x) & -\sin(x) & e^x \\ -\sin(x) & -\cos(x) & e^x \end{bmatrix}$

→ they are linearly Independent!

→ $\begin{bmatrix} + & + & + \\ + & - & + \\ - & - & + \end{bmatrix}$ → there is no constant any column can be multiplied by & added to any other to make it equal to another!

Let $V_1 = 2 + 3x + x^2$
$V_2 = 1 + x$
$V_3 = 1 + 3x + x^2$

$\rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. (10 points) Let $A, B, \& C \in M_{n\times n}(\mathbb{R})$ be such that C is an invertible matrix & $AB = C^3$, show that B is invertible
↳ recall a matrix m is invertible if $\det(M) \neq 0$
Thus $\det(C) = 0$ meaning that $\det(C^2)$ & $\det(C^3) \neq 0$
if $\det(A)$ or $\det(B) = 0$ then $\det(C^3) = 0$, since the only number that cubes to zero is zero if $\det(C) \neq 0$ then $\det(B) \neq 0$ meaning it is invertible!

2. (10 Points) Given that $\det\left(\begin{bmatrix} 2a+d & 2b+e & 2c+f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 1$ Find

$2A + \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$ The determinant of A^* is 2 since it is multiplied by a scalar and then -2 since the row order is permutated

3. (20 Points) Determine if the following are valid subspaces
a. $V = \mathbb{R}^4$ & $W = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_2 = x_3 + x_4\}$
Zero vector?
↳ $(x_1, x_2, x_3, x_4) = (0,0,0,0)$ $x_1 + x_2 = x_3 + x_4 \rightarrow 0 = 0$ ✓
Closure?
Scalar multiplication → $(2,3,4,1)(-1)$
$2+3 = 4+1$ ✓ $-2-3 = -4-1$ ✗
Not a subspace! fails closure under scalar!

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$V = P_4(\mathbb{R})$ & $W = \{p \in V \mid p(1) > p(2) = 2p(3)\}$

$ax^4 + bx^3 + cx^2 + dx + e$

→ $p(1) = a+b+c+d+e$

$p(2) = 16a + 8b + 4c + 2d + e$

$p(3) = 81a + 27b + 9c + 3d + e \rightarrow 2p(3) = 162a + 54b + 18c + 6d + 2e$

↳ $(a+b+c+d+e)(16a+8b+4c+2d+e) = 162a + 54b + 18c + 6d + e$

$16a + 8b + 4c + 2d + e = 162 + 54 + 18 + 6 + 1$

$16a + 8b + 4c + 2d + e = 241$

↳ Fails zero vector!

4. (10 Points) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ $A_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Determine whether $A \in \text{span}\{A_1, A_2, A_3\}$ or not if so express A as a linear combo of $A_1, A_2, \& A_3$

$A_1 + A_2 + A_3 ? = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} - A_2 \rightarrow \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

$1X_1 \quad 0X_2$

$1X_1 \quad 0$

$0 +$

$X_1 + 0 + 0 + X_4$

$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

$X_1 + X_2 + 0 + X_4$

$X_1 + 0 + X_3 + 2$

$0 - X_1 + 0 - X_3 + 0$

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5. (10 points) $V = M_{3\times2}(\mathbb{R})$ & $S = \{A \in V \mid a_{11} + a_{12} = 3a_{12} + 2a_{32}\}$
find a spanning set for S

$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ $a+d = 3b+2f$

e & C are free!

$z = 3b+2f$

$\begin{bmatrix} & \\ & \\ & \end{bmatrix}$

1 1
3 2

6. (10 points): Is the set $\{x^3 - x,\ 2x^2 + 4,\ -2x^3 + 3x^2 + 2x + 6\}$

$x_3 + \emptyset - x + \emptyset$

$-2x^3 + 2x^2 + \emptyset + 4$

$\emptyset + 3x^2 + 2 + 6$

NO!

$\begin{bmatrix} 1 & \emptyset & -1 & \emptyset \\ -2 & 2 & +\emptyset & 4 \\ \emptyset & 3 & 2 & 6 \end{bmatrix}$

$R_3 + R_2 + 2R_1$

$[0,3,2,6] + [-2,2,0,4] + 2[1\ 0\ -1\ 0]$

$= [0,5,0,10]$

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Vector Space Maps:

Let V & W be vector spaces, A vector space map, T is is a rule/function, that assigns to each $v \in V$ a vector $w \in W \longrightarrow$ i.e. $T(v_1) = w$

↳ Example: if A is a $m \times n$ matrix $V = \mathbb{R}^n$ $W = \mathbb{R}^m$ then, $T: V \to W$ defined by $T(x) = Ax (\in \mathbb{R}^m = W)$ is a vector space map from $\mathbb{R}^n \to \mathbb{R}^m$

A map $T: V \to W$ is called a Linear Transformation (LT) if ↱ 1. $T(v_1 + v_2) = T(v_1) + T(v_2) \to$ LTs preserve vector addition
↳ 2. $T(cv) = cT(v) \longrightarrow$ LTs preserve scalar multiplication
This means all linear combinations in V have corresponding linear combinations in W

Example: Define the vector space map $T: C^2(I) \to C^0(I)$ by $T(y) = y'' + y$
Verify that T is a linear transportation
↳ Suppose $y_1, y_2 \in C^2(I)$

$$\begin{aligned} T(y_1 + y_2) &= (y_1 + y_2)'' + (y_1 + y_2) \\ &= y_1''' + y_2'' + y_1 + y_2 \\ &= (y_1'' + y_1) + (y_2''' + y_2) \\ &= T(y_1) + T(y_2) \end{aligned}$$ ✓

$$\begin{aligned} T(cy) &= (cy_1)'' + cy_1 \\ &= cy'' + cy_1 \\ &= c(y'' + y) \\ &= cT(y) \end{aligned}$$ ✓

→ Thus, T is a LT!

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Rank Nullity & Systems of Linear Equations:

↳ Let A be an $m \times n$ matrix & consider the homogenous system $Ax = 0$

1. If $\text{rank}(A) = n$, then, since $\text{rank}(A) + \text{nullity}(A) = n$
   ↳ $\text{nullity}(A) = 0$
   ↳ $\text{nullspace}(A) = \{0\}$
   ↳ $Ax = 0$ has only the trivial solution
2. If $\text{rank}(A) = r < n$, then $\text{nullity}(A) = n - r$ so, there are $n - r$ general solutions to the system

Now... Suppose $Ax = b$ is a nonhomogenous vector equation Let x_p be any solution to $Ax = b$, which we will call a particular solution & suppose $\{x_1, \cdots, x_{n-r}\}$ is a basis for nullspace(A). Then any solution to $Ax = b$ is of the form

$$X = c_1x_1 + c_2x_2 + \cdots + c_{n-r}x_{n-r} = x_c + x_p$$

→ any solution to a nonhomogenous linear system looks like a solution to a corresponding homogenous linear system plus a particular solution to $Ax = b$

Example: Find the general solution to the linear system

$$A = \begin{bmatrix} 2 & 6 & -8 \\ -1 & 3 & 4 \\ 5 & -15 & -20 \\ -2 & 6 & 8 \end{bmatrix} \quad b = \begin{bmatrix} -18 \\ 9 \\ -45 \\ 18 \end{bmatrix}$$

$Ax = b$

Recall: The general solution is of the form: $X = X_p + X_0$ ← The general sol. to $Ax = 0$

→ Verify by finding the solution set as we normally would

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ RREF of A, x_2 & x_3 are free variables

Let $c_1 = x_2$ & $c_2 = x_3$ → Basis for null space

$$X = \begin{bmatrix} -9 + 3c_1 + 4c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Particular sol.

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Properties of Linear Transformations:
↳ Recall: a transformation is linear if it preserves closure under vector addition & scalar multiplication
1. LTs send zero vectors to zero vectors (ie $T(\emptyset v) \to \emptyset w$)
2. LTs send inverses to inverses (ie $T(-v) = -T(v)$)

Let A be an $m \times n$ real matrix then $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$ is a linear transformation
↳ this means any LT can be represented as a matrix

6.3: Kernel & Range of Linear Transformations
Let $T: V \to W$ be an LT, then the set of vectors $v \in V$ such that $T(v) \to 0$ is called the kernel of T (kern(T))
↳ $kern(T) = \{v \in V \mid T(v) = \emptyset w$
↳ When $T(x) = Ax$ then $kern(T) = nullspace(A)$ & $Kern(T)$ is a subspace of $\mathbb{R}^n$ where T is defined
. The set of all vectors $w \in W$ such that there exist $v \in V$ where $T(v) = w$ is called the range of T (Rng(T))

General Rank Nullity Theorem:
↳ If $T: V \to W$ is a LT and V is a finite dimensional vector space then, $\dim(kern(T)) + \dim(Rng(T)) = \dim(V)$

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MATH 165 Written HW:

1. For each part below determine if the given mapping T is linear or not

a. $T: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = AB - BA$ where B is a fix-n×n matrix

↳ Recall: a transformation is linear if it preserves scalar multiplication & vector addition:

multiplication: let c be an arbitrary scalar

↳ for $T(A)$ to be linear $cT(A) = T(cA)$

$AB - BA + CB - BC \to c(AB - BA) = (cA)B - B(cA)$

$= T(A) + T(C)$ LINEAR! → preserve closure under scalar

b. $M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = A^2$

$M_2(\mathbb{R}) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ → Multiplication: $cT(A) = T(cA)$

$T(A) = A^2$ $T(cA) = (cA)^2$

$cT(A) = c(A)^2$ ✗

NOT LINEAR!

Fails to preserve scalar multiplication!

c. $T: M_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$

↳ multiplication

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $ad - bc = \det(A)$ $cT(A) = T(cA)$

$c\det(A) = \det(cA)$ ✓

↳ Addition: $T(A_1 + A_2) = T(A_1) + T(A_2)$

$A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $T(A_1) = \det(A_1) = ad - bc$ $T(A_1 + A_2) = (a+e)(d+h) - (b+g)(c+d)$ ✓

$A_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ $T(A_2) = \det(A_2) = eh - fg$

→ Linear!

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2a. Let $T_A: \mathbb{R}^4 \to \mathbb{R}^3$ be the matrix transformation corresponding to the matrix. Find a basis for the kernel & range of A

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Recall: $\text{kern}(A) = (T(A) = \emptyset)$

$Tx = Ax$ then $\text{kern}(T) = \text{Nullspace}(A)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$X_1 - X_4 = \emptyset$ $\quad X_1 = X_4$

$X_2 + X_4 = \emptyset$ $\quad X_2 + X_4 = X_3 + X_4$ basis kern A

$X_3 + X_4 = \emptyset$ $\quad = [1\,0\,1\,0], [0,1,0,1]$

$$\text{Range}(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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Complex Numbers

↳ Complex numbers are those of the form $z = a + bi$ where i represents the imaginary number obtained by taking $\sqrt{-1}$

↳ Euler's Identity: for every $X \in \mathbb{R}$

$e^{ix} = \cos(X) + i\sin(X)$

(this can be proven using Taylor Series)

7.1: Eigen value/Eigen vectors

↳ (Also called the characteristic values/vectors)

Let A be an $m \times m$ matrix, any values λ for which $Av = \lambda v$ has non-trivial solutions ($v \neq \emptyset$) are the eigenvalues of A, the corresponding vectors are the eigenvectors of A

We will need to consider A as the matrix of a linear transformation from $C^r \to C^n$

consider $A = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix}$ & $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Av_1 = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix} = -7v_1$ → -7 is an eigenvalue of A w/ eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solution to the eigenvalue/eigenvector problem:

If A is an $n \times n$ matrix $Av = \lambda v$ iff $Av - \lambda v = 0$

then $(A - \lambda I)v = 0$ has nonzero solutions when the $\det(A - \lambda I) = 0$

↳ $\det(A - \lambda I)$ is a polynomial of degree n w/ variable λ & is called the charecteristic polynomial of A

↳ the solution to the charactaristic polynomial are the eigen values of A

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The Process for Finding Eigenvalues:

1. Find all scalars λ_i such that $\det(A-\lambda_i I)=0$
2. If $\lambda_1,\cdots,\lambda_k$ are the distinct eigenvalues, then solve the k linear systems $(A-\lambda_i I)v=0$ for $i=1,\cdots,k$ to find the corresponding eigenvectors for each eigenvalue

Proposition: An $n\times n$ matrix A is invertible if & only if 0 is not an eigenvalue of A

↳ If 0 is an eigenvalue, then there is some $v\neq 0$ such that $Av=0v=0 \rightarrow A$ is not Invertiable

Example:

$A=\begin{bmatrix}3 & -1\\ -5 & -1\end{bmatrix} \rightarrow (A-\lambda I)=\begin{bmatrix}3-\lambda & -1\\ -5 & -1-\lambda\end{bmatrix}$

$\det(A-\lambda I)=(3-\lambda)(-1-\lambda)-5$

$=\lambda^2-2\lambda-8$

$=(\lambda-4)(\lambda+2)\rightarrow$ thus -2 & 4 are eigen values

Solve $(A-4I)v=0$

$\begin{bmatrix}-1 & -1 & 0\\ -5 & -5 & 0\end{bmatrix}\rightarrow\begin{bmatrix}-1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\rightarrow\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ $x_2=5$, $x_1=-5$

eigenvectors $=\{\begin{bmatrix}-5\\ 5\end{bmatrix}\mid s\in\mathbb{R}\}$ & $\{\begin{bmatrix}-1\\ 1\end{bmatrix}\mid s\in\mathbb{R}\}$

Solve $(A+2I)v=0$

$\begin{bmatrix}5 & -1 & 0\\ -5 & 1 & 0\end{bmatrix}\rightarrow\begin{bmatrix}5 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\rightarrow\begin{bmatrix}1 & -1/5 & 0\\ 0 & 0 & 0\end{bmatrix}$ $x_2=5$, $x_1=1/5\,s$

eigenvectors: $\{\begin{bmatrix}1/5\\ 1\end{bmatrix}s\mid s\in\mathbb{R}\}$

↳ Notice $\{\begin{bmatrix}1/5\\ 1\end{bmatrix},\begin{bmatrix}-1\\ 1\end{bmatrix}\}$ are linearly Independent & correspond to different eigen values

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Eigenvectors Example #2: Let $A=\begin{bmatrix}5 & 12 & -6\\ -3 & -10 & 6\\ -3 & -12 & 8\end{bmatrix}$
Find the eigenvectors of A

↳ $\det(A-\lambda I)=0$

$$\begin{vmatrix}(5-\lambda) & 12 & -6\\ -3 & (-10-\lambda) & 6\\ -3 & -12 & (8-\lambda)\end{vmatrix}=0 \rightarrow (\lambda-2)^2(\lambda+1)=0$$

$\lambda_1=2$ (w/ multiplicity 2)
$\lambda_2=-1$

Solve $(A-2I)v=0$

$$\begin{bmatrix}3 & 12 & -6 & 0\\ -3 & -12 & 6 & 0\\ -3 & -12 & 6 & 0\end{bmatrix} \rightarrow \begin{bmatrix}3 & 12 & -6 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 4 & -2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$x_2=s$ & $x_3=t$ $\begin{bmatrix}1 & 4 & -2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

eigenvectors ⟶

$$\left\{\begin{bmatrix}-4s+2t\\ s\\ t\end{bmatrix} \mid s,t\in\mathbb{R}\right\}$$

Example #3: Let $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ $p(\lambda)=\det(A-\lambda I)$
↑ chactaristic Polynomial

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 1\\ 0 & 1-\lambda\end{vmatrix}=(1-\lambda)^2=0$$ ⟵ Charecteristic equation

$\lambda_1=1$ (multiplicity 2)

If A is an $n\times n$ matrix, $p(\lambda)$ will have degree n
↳ Solve $(A-I)v=0$

$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ = eigenvectors corresponding to $\lambda_1=1$ are all the complex conjugate of λ given by $a-bi$. Similarily, if $v\in C^n$ where $v=\begin{bmatrix}a_1+b_1i\\ a_2+b_2i\\ \vdots\\ a_n+b_ni\end{bmatrix}$

Thus the complex conjugate of λ is
↳ $\begin{bmatrix}a_1-b_1i\\ a_2-b_2i\\ \vdots\\ a_n-b_ni\end{bmatrix}$

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Let A be an n×n matrix w/ real entries
If λ is a complex eigenvalue of A w/ eigenvector of $\bar{v}$
Then $\bar{\lambda}$ is also an eigen value of A & has an eigenvector $\bar{v}$

Let $A=\begin{bmatrix} 9 & 37 \\ -1 & -3 \end{bmatrix}$ $P(\lambda)=\det(\lambda-\lambda I)=\begin{vmatrix} 9-\lambda & 37 \\ -1 & -3-\lambda \end{vmatrix}$

$= (9-\lambda)(-3-\lambda)+37$

$= \lambda^2-6\lambda+10$

$\left[\begin{array}{cc|c} 9-(3+i) & 37 & 0 \\ -1 & -3-(3+i) & 0 \end{array}\right]$ $R_1=(-6+i)R_2$

→ We just want to solve $(A-(3+i)I)v=0$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 6+i & 0 \end{array}\right]$ $x_2=s$, $x_1=-(6+i)s$

eigenvectors $= \text{span}\left\{\begin{bmatrix} -(6+i) \\ 1 \end{bmatrix}\right\}$

→ Every eigenvector is a multiple of $\begin{bmatrix} -(6+i) \\ 1 \end{bmatrix}=\begin{bmatrix} -6-i \\ 1 \end{bmatrix}$

If A is a 3x3, then P(λ) has degree 3, so if there is a complex root then P(λ) has a complex root, its conjugate, and a real root

7.2: General Results for eigenvectors & eigenvalues

→ $P(\lambda)=\det(A-\lambda I)$ ← Characteristic polynomial

→ $P(\lambda)=(-1)^n\lambda^n+b_1\lambda^{n-1}+b_2\lambda^{n-2}+\cdots+b_n$

$=(-1)^n(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}$

Where $\lambda_1,\ldots,\lambda_k$ are the distinct eigenvalues of A

& $n_1+n_2+\cdots+n_k=n$

→ n_i is called the algebreic multiplicity of λ_i

→ Let A be an n×n matrix w/ real entries & let E_i denote the collection of all vectors $v\in\mathbb{C}^n$ such that $Av=\lambda_i v$
Then, E_i is called the Eigenspace corresponding to the eigenvalue λ_i

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Why is hell

Hell

keep

→ If $T: C^n \to C^n$ given by $T(v) = (A - \lambda_i I)v$, then $E_i = \ker(T)$

→ The eigenspace E_i is a subspace of C^n for each i

Note: the eigenspace E_i is slightly different than the set of all eigenvectors corresponding to λ_i

↳ determine the eigenspace for

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \quad P(\lambda) = (\lambda-2)^2(\lambda+1)$$

$\lambda_1 = 2 \to V_1 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\lambda_2 = -1$ $V_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Properties of Eigenvalues:

↳ exponents on the eigenvalue

Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$ Find its eigenvectors & eigenvalues

↳ $\det(\lambda I - A) = 0$

↳ $\det\begin{bmatrix} \lambda+5 & -2 \\ 7 & \lambda-4 \end{bmatrix} = 0$ $a'c - bd = 0$

$(\lambda+5)(\lambda-4) + 14 = 0$

This is the charectaristic → $\lambda^2 + \lambda - 6 = 0$

Polynomial of A, the roots of which are the eigenvalues of A → $\lambda_1 = 2$ & $\lambda_2 = -3$

$\lambda_1 = 2$

↳ $\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As an aug matrix

$\begin{bmatrix} 7 & -2 & | & 0 \\ 7 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{7} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} = s\begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix}$ → To verify this as an eigen-vector ensure $AX = \lambda X$

$\lambda_1 = 2 \to 2\begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \to \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ ✓

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Yet another Eigenvector Example:

Let $A = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$ → Find the Eigenvalues/Eigenvectors of A:

Step #1: Solve $\det(\lambda I - A) = 0$

↳ $\det(\lambda - A) = 0 \longrightarrow \det\begin{bmatrix} \lambda-5 & 10 & 5 \\ -2 & \lambda-14 & -2 \\ 4 & 8 & \lambda-6 \end{bmatrix} = 0$ → Use cofactor expansion to solve

$(\lambda-5)(\lambda^2-20\lambda+100) = 0$

$(\lambda-5)(\lambda-10)^2 = 0 \rightarrow \lambda_1 = 5$, & $\lambda_2 = 10$ (multiplicity = 2)

Thus, we have the two eigenvalues of A which we can use these eigenvalues then, to calculate the eigenvectors of A by solving $(\lambda I - A)X = 0$ (only nonzero sols.)

$\lambda_1 = 5 \rightarrow (5I - A)X = 0 = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

↳ Put into Aug. Matrix

$\begin{bmatrix} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{bmatrix}$ RREF! $\begin{bmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → Thus, any vector of the form $\begin{bmatrix} \frac{5}{4}s \\ -\frac{1}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

$x_1 = \frac{5}{4}$

$x_2 = -\frac{1}{2}$

$X_1 = t\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$ ← We can multiply this equation by 4 to get a nicer representation ← Is an eigenvector!

↳ To check that this is a valid eigenvector we can multiply the original matrix by it & verify that it solves to $5X_1$

↳ Where 5 is the corresponding λ_1

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Eigenvectors Continued:

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \quad P(\lambda) = \det(A - \lambda I) = (\lambda - 2)^2(\lambda + 1)^1 \quad \rightarrow \begin{bmatrix} 5-\lambda & 12 & -6 \\ -3 & -10-\lambda & 6 \\ -3 & -12 & 8-\lambda \end{bmatrix}$$

Chachacteristic Polynomial

Eigenvalues:

$\lambda_1 = 2$ (w/ multiplicity 2)

$\lambda_2 = -1$ (w/ multiplicity 1)

Eigenvectors:

$\lambda_1 = 2 \rightarrow V_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1 \rightarrow V_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Recall: The **Eigenspace** of a matrix A w/ real entries that has the Eigenvalues of $\{\lambda_1, \cdots, \lambda_k\}$ with the corresponding Eigenvectors $V_1, \cdots, V_k$ is denoted as (for λi)

$E_i = \text{span}\{V_1, \cdots, V_k\}$ → this set contains all vectors that satisfy $Av = \lambda_i V$

↳ (including the 0 vector)

↳ E_i is a subspace of $\mathbb{C}^n$

↑ this denotes the set of all n-dimensional complex numbers

Note: Each Eigenvalue has its own eigenspace

Additionally: $E_i = \text{kern}(A - \lambda_i I)$

If the Eigenvectors of a $n \times n$ matrix A are linearly independent then we can say A is non-defective, otherwise A is said to be defective

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MATH 165 HW 10

1a. suppose that an $n \times n$ matrix A satisfies $\sum_{j=1}^{n} a_{ij} = r$
That is, the sum of the entries in each row is r
Show that r is an eigenvalue of A & determine an eigenvector → Recall: an eigenvalue of a matrix is any value λ such that $Av = \lambda v$ → $\det(A - \lambda I)$

~~Let A be a 2x2 matrix~~ ~~$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$~~

If A is a 2×2 matrix → $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$a + c = r$

$b + d = r \rightarrow a + c - b + d$

↳ If r is an eigenvalue it must satisfy $Av = \lambda v$

$\det(A - rI)$

$= \begin{vmatrix} a-r & c \\ b & d-r \end{vmatrix}$ $\begin{cases} (a-r)(d-r) - bc = 0 \\ \text{~~} \\ \hookrightarrow r = a + c \quad (a+a-c)(d-b+d) - bc = 0 \\ r = b + d \quad (-c)(-b) - bc = 0 \\ bc - bc = 0 \checkmark \end{cases}$

↳ Thus r must be an eigenvalue

an eigen vector then would be any $\begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix}$

$A = r \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = r \begin{bmatrix} d \\ e \end{bmatrix}$

1b. Prove that the eigenvalues of an upper triangular matrix are just the diagonal entries:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow P(\lambda) = \det(A - \lambda I)$

$\det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 3-\lambda \end{bmatrix} \rightarrow (1-\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 0 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 \\ 0 & 3-\lambda \end{vmatrix} + 3 \begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$

$(1-\lambda)(2-\lambda)(3-\lambda) - 0 - 0 - 0 \quad \leftarrow$

↳ the charectaristic polynomial is only zero when λ = an entry of the main diagonal

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7.2 General Results for eigenvalues/vectors

↳ If A is a nxn real matrix, then the charecteristic polynomial is given by $P(\lambda) = \det(A - \lambda I)$ or $= (-1)^m (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$ where m is the algebraic multiplicity of λ_i

↳ $m = m_1 + m_2 + \cdots + m_k$

Let $T_i: C^n \to C^n$ be the linear transformation given by $T(v) = (A - \lambda_i I) \to$ Then, the eigenspace E_i - Kern(T)

NOTE: the zero-vector is never an eigenvector but it is always in the Eigenspace

Let λ_i be an eigenvalue of A w/ multiplicity n_i & let the corresponding eigenspace E_i be:

1. A subspace of C^n
2. If $d_i = \dim(E_i)$ Then $1 \leq d_i \leq n_i$

Diagonolization:

Let A & B be n×n matricies, they can be called similar if there is an invertible matrix S such that

$A = S^{-1} B S$

If A is a n×n matrix w/ eigenvectors $v_1, \ldots, v_n$ & A is non-defective then A is similar to a diagonol matrix

Diagonalization: Let A & B be matricies (n×n)
↳ A & B are said to be similar if there is an invertible matrix S such that $A = S^{-1}BS$

If A is a n×n non-defective matrix w/ eigenvalues $\lambda_1 \cdots \lambda_n$ & corresponding eigenvectors $v_1, \cdots, v_n$
↳ Then, A is similar to a diagonal matrix

Let $S = [v_1, v_2, \cdots, v_n]$ ← column vectors, if $v_1, v_2, \cdots, v_n$ are LI
→ Rank(S) = n ⟷
→ S is invertable

$S^{-1}AS = S^{-1}A[v_1 \cdots v_n]$
$= S^{-1}[Av_1 \cdots Av_n]$ ← since $S^{-1}S = I$, $S^{-1}v_i$ is the n^{th} column vector of I (a vector with a 1 in the i^{th} spot & zeros otherwise)

$$S^{-1}AS = [v_1 \cdots v_n] - \begin{bmatrix} \lambda_1 & \emptyset & \cdots \\ \emptyset & \lambda_2 & \cdots \\ \emptyset & \emptyset & \\ \vdots & \vdots & \end{bmatrix}$$

Suppose T(x) = A(x) represents some important LT & we want to compute $A^{40}(x)$
If $S^{-1}AS = \text{diag}(\lambda_1, \cdots, \lambda_n) = D$, then $A = SDS^{-1}$
↳ $A^{40} = (SDS^{-1})^{40}$
$= (SDS^{-1})(SDS^{-1}) \cdots$
$= SD^{40}S^{-1} \rightarrow$

$$\begin{bmatrix} \lambda_1^{40} & \cdots & \\ \vdots & \lambda_2^{40} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

The process of finding S such that $S^{-1}AS = D$ is called diagonolization
↳ Every symetric matrix is diagonozable

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Diagonalization Example:

$$A=\begin{bmatrix} 5 & 12 & -6 \\ -3 & 10 & 6 \\ 8 & -12 & 8 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = -1$

vectors: $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Let $S = [v_1, v_2, v_3]$

$$= \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \text{Then, } S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

note: Diagonalization is not unique!

↳ $S = [v_3, v_1, v_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ → both are valid!

8.1: General Theory for Linear Differential equations

Consider the mapping $D: C'(I) \rightarrow C^0(I)$ given by $D(f) = f'$

↳ D is a linear transformation (preservers scalar mult & vector addition)

↳ In general, $C^k(I) \rightarrow C^{k-1}(I)$ is a LT for $k \geq 1$

D is called the derivative operator

Taking linear combinations of these derivative operators gives the general linear differential operator of order n

$$L = a_1(x)D^n + a_2(x)D^{n-1} + \cdots + a_0(x)I$$

Where ai are general functions of the variable x, defined by

$$Ly = a_1(x)y^n + a_2(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y$$

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Example: Let $L = D^2 + 4xD - 3x$

Then, $L(y) = y'' + (4x)y' - (3x)y$

↳ $L(\sin x) = -\sin x + 4x\cos x - 3x\sin x$

Example #2: Let $L = D - 2x$ Find $\ker(L)$

↳ $\ker(L) = \{ y \in C^1(I) \mid L(y) = 0 \}$

$= \{ y \mid y' - 2xy = 0 \}$

Notice $\ker(L)$ is equivalent to the solution set for the 1st order LDE $y' - 2xy = 0$

We can solve using integrating factors

can

are y ourselves

Hello

8.2: Constant Coefficient Linear Differential Equations

↳ How can we solve LDEs of the form

$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = F(x)$ is continious on I

We can put them in the form $Ly = F(x)$

where $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$

↑ L is called the polynomial differential operator

To find the solution we need to consider the homogenous polynomial differential operator → $Ly = 0$

The auxilary polynomial can be obtained by plugging in $P(D)y = 0 \rightarrow y = e^{kx} \rightarrow$

$(k^n + h_1 k^{n-1} + \cdots + q$

Auxilary Polynomials:

$y = (C_1 + C_2 x + C_3 x^2 + \cdots C_n) e^{mx}$

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8.2: Constant-Coefficient Homogenous DEs

$y'' + 16y = 0 \quad y(\frac{\pi}{2}) = -10 \quad y'(\frac{\pi}{2}) = 3$

$r^2 + 16 = P(r)$

roots: $\pm 4i$

$\cos(4t) \quad \sin(4t)$

$y(t) = c_1\cos(4t) + c_2\sin(4t)$

$y'(t) = -4c_1\sin(4t) + 4c_2\cos(4t)$

$$\left[\begin{array}{cc|c} \cos(4t) & \sin(4t) & -10 \\ 4\cos(4t) & 4\sin(4t) & 3 \end{array}\right]$$

$c_1 = -10$ & $c_2 = 3/4$

$y(t) = -10\cos(4t) + \frac{3}{4}\sin(4t)$

Solve the Initial value problem:

1. Begin by finding the charecteristic polynomial & its roots
2. Convert the complex roots into their corresponding trigonometric forms & find the general solution
3. Take the derivative of the general solution to set up a system of equations to solve for the coefficients
4. Plug in the initial values into the system & solve for the coefficients
5. Plug in the new coefficients into the general solution to give the particular solution for the given conditions

How to deal with complex numbers?

↳ Euler's Identity: For every $x \in \mathbb{R}$

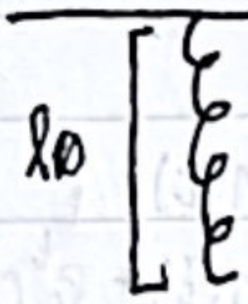
$$e^{ix} = \cos(x) + i\sin(x)$$

When you have $\pm ai$ as roots Hn

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Application: Oscillators of mechanical systems

A spring is suspended in space with a nautral length of l_0



Imagine we attatch an object with mass & let it come to rest at what we will call the equilibrium position



We pull the mass down & release it with initial velocity & let the mass oscillate up & down

Let $y(t)$ be the distance the spring is streched past the equilibrium position

We consider forces acting on the mass:

1. Gravity, $F_g = mg$
2. Spring force, $F_s = -k(L_0 + y(t))$ ⟶
3. An external dampening force $F(d)$
   ↳ comes from things like air res.
   $F_d = -L\, dy/dt$

> Note: ↓ is positive
> ↑ is negative
> the spring constant $k > 0$

By Newton's 2nd law

$ma(t) = F_g + F_s + F_d$

$a(t) = y''$

$my'' = mg - k(L_0 + y) - cy' \rightarrow mg = kL_0$

$my'' = kL_0 - k(L_0 + y) - cy'$

We can consider the case where we introduce an external "driving force" $F(t)$ which acts on the mass

$my'' = kL_0 - k(L_0 + y) - cy' + F(t)$

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Thus, we have a second order linear differential constant coefficient DE with initial conditions

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t) \qquad y(0) = y_0 \quad y'(0) = v_0$$

↳ When there is no driving force & is dampening

$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$ → Aux. Polynomial

$$P(r) = r^2 + \frac{c}{m}r + \frac{k}{m}$$

Roots of $P(r)$: $\frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

Let $\Delta = c^2 - 4km$ be the discriminant

We will examine:

1. $c^2 - 4km = \Delta < 0$ (complex roots)

↳ No external force & no dampening → Simple Harmonic Motion

Aux. Polynomial: $P(r) = r^2 + \frac{k}{m}$

roots: $\pm i\sqrt{k/m}$

gen. solution: $y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{k/m}\, t\right)$

↑ This form is difficult to interpret. We can rewrite this solution using trigonometry

$A_0 = \sqrt{c_1^2 + c_1^2}$ $\qquad W_0 = \sqrt{\frac{k}{m}}$

$\Phi = \arctan\left(\frac{c_2}{c_1}\right)$

↳ $y(t) = \underline{A_0} \cos(W_0 t - \Phi)$

↳ the amplitude of motion

W_0 is called the circular frequency

$T = \frac{2\pi}{W_0}$ → the period $\qquad \frac{1}{T}$ → the frequency

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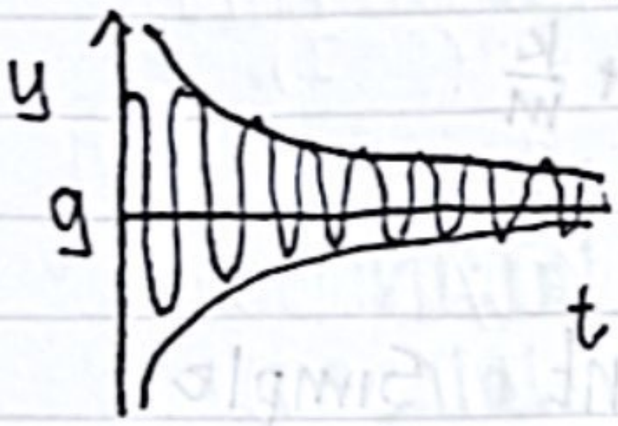
Next we consider dampening ($C \neq 0$)

$\Delta = C^2 - 4km$ as $\Delta = C^2 - 4km < 0$ complex roots

→ Underdampening

General Solution:

$\underbrace{A_0 e^{-Ct/2m}}_{\text{↳ as } t \to \infty} \cos(\mu t) - \phi$ Where $\mu = \frac{\sqrt{C^2 - 44cm}}{2m}$

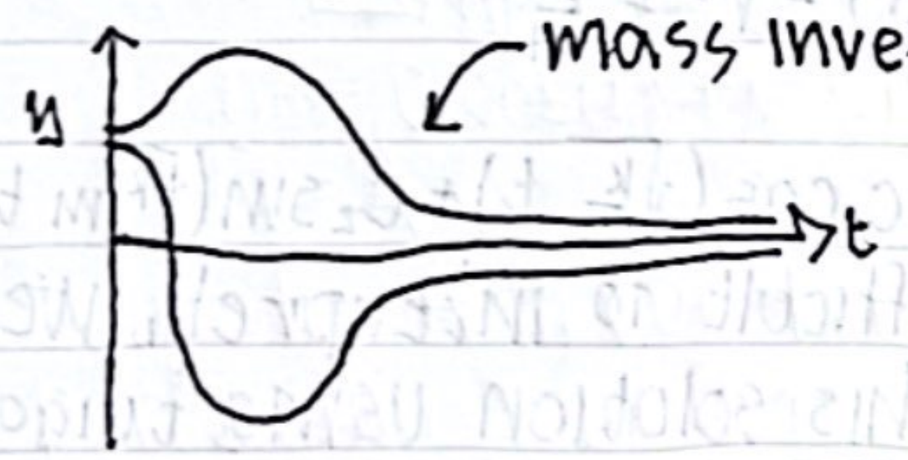


← the oscilations get smaller over time

Case #2: $\Delta = 0$ Critical Dampening

↳ repeated real root

$y(t) = e^{\frac{-ct}{2m}} + t c_1 e^{\mu t} + c_2 e^{-\mu t}$



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Annihilator Example: Find the general

$y'' + y = 8e^{2t}$

1. solve $(D^2+1)y = 0$
   ↳ roots $= \pm i$
   $y_c(t) = c_1\cos(t) + c_2\sin(t)$
2. Find an annihilator, A(D), for $8e^{2t}$
   $(D-2)\,8e^{2t} = 0$
   $A(D) = (D-2)$
3. Solve $A(D)(D^2+1)y = 0$
   $(D-2)(D^2+1) = 0$
   $= A_1e^{2t} + c_1\cos(t) + c_2\sin(t)$
   ↳ Note: A denotes a scalar from the annihilator
   $= y_c + y_p$ → y_p is our trial solution
   ↳ general solution to $(D^2+1)y = 0$
   $y_c = c_1\cos(t) + c_2\sin(t)$
   $y_p = A_1e^{2t}$
4. Solve for A in our trial solution to get an actual particular solution
   $(D^2+1)y_p = 8e^{2t}$
   $(D^2+1)Ae^{2t} = 8e^{2t}$
   $D^2Ae^{2t} + Ae^{2t} = 8e^{2t}$
   $4Ae^{2t} + Ae^{2t} = 8e^{2t}$
   $5Ae^{2t} = 8e^{2t}$
   $A = 8/5 \longrightarrow y_p = 8/5\,e^{2t}$

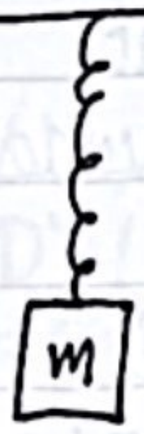
General Solution: $y_c + y_p = c_1\cos(t) + c_2\sin(t) + \frac{8}{5}e^{2t}$
↳ to find the particular solution plug in the given values & solve for the constants

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Recall, by Newtons laws (Apparently)...

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$$

#3.



$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + y = 0$$

When is the system underdampened?

Let:
- m = mass of the object
- α = the spring constant
- F_d = the external dampening force
  - ↳ $-L\frac{dy}{dt}$

Initial Conditions:
- $y'(0) = 0$

When $2\alpha = c/m$ & $c^2 - 4km = \Delta < 0$ ←

If $\Delta < 0$ then the charecteristic Polynomial only has complex roots

↳ Simple Harmonic motion:
No external force & No dampening

When is the system critically dampened? → When $\Delta = 0$

The charecteristic polynomial has a repeated real root

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Recall:

$$y'' + \frac{c}{m}y' + \frac{K}{m}y = \frac{F(t)}{m}$$

C = dampening coefficient
m = mass of the object
k = spring constant
F(t) = external driving force

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + y = 0$$

When $2\alpha = \frac{c}{m}$ & $C = 0$
↳ underdampened!

When $\Delta = c^2 - 4km = 0$ → Criticallu Dampened

When $\Delta = c^2 - 4km < 0$ & $\Delta > 0$ → Overdampened!
↳ roots of the chavecteristic Polynomial: $= \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4}}{2}$

General Solution:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$y'(t) = r_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}$$

$y'(0) = 0$

$$0 = r_1 + r_2$$

*since both roots are negative this mass will never pass equilibrium, rather as $t \to \infty$ $y(t)$ goes to equitibium, but never crosses

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9.1: First Order Linear Systems:

↳ A system of equations is considered First order linear if the unknown functions can be written of the form $x_1(t), \cdots, x_n(t)$

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1(t)$$
$$x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2(t)$$
$$\vdots \qquad\qquad \vdots$$
$$x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots a_{nn}x_n(t) + b_n(t)$$

Every higher order linear system can be written as a first order linear system

Example: $x'' + 4e^t x' - 9t^2 x = 7t^2$

Let $y = x'$

$$x' = (7t^2 + 9t^2 x) \rightarrow x' = \frac{7t^2}{4e^t} + \frac{9t^2}{4e^t}x$$

Vector Formulation

$$X(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \frac{dx}{dt} = X'(t) = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \qquad b = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

Then our 1st order system can be rewritten

$$x'(t) = A(t)x(t) + b(t)$$

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Theorems of First Order Linear Systems

1. Let $V_n(I)$ denote the set of all column vectors of functions on the interval I. Then $V_n(I)$ is a vector space
2. Let $x_1(t), \dots, x_n(t)$ be vectors in $V_n(I)$ then their Wronksian is the determinant of the matrix $W(t) = [x_1(t)\ x_2(t) \cdots x_n(t)]$
   ↳ If the wronksian is non-zero for some $t \in I$ then x is linearly Independent
3. Let A(t) be the matrix for the system. If $x_1(t), \dots, x_n(t)$ is a linearly Independent set of solutions to $x_i = A(t)x$ on I then their wroksian is non-zero at every $t \in I$

9.3: General Results for First Order Linear Systems

Consider the Initial Value Problem

$x'(t) = A(t)x(t) + b(t)$ → Where A(t) & b(t) consist of continious functions the IVP has the unique solution I

$x(t_0) = x_0 = \begin{bmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix}$

The set of solutions to the homogenous system $b(t) = \emptyset$ $x'(t) = A(t)x(t)$ is a subspace of $V_n(I)$ with dimension equal to n

Any n linearly Independent solutions $\{x_1, \dots, x_n\}$ to $x' = A(t)x$ is called a fundemental solution set on I
↳ The matrix $X(t) = [x_1(t) \cdots x_n(t)]$ is called the fundemental matrix

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The Non-Homogenous Case: $b(t) \neq 0$

Let $\{x_1, \dots, x_n\}$ be a fundemental solution set to $x'(t) = A(t)x(t)$ & $x_p(t)$ be any particular solution to the system $\longrightarrow x_p'(t) = A(t)x_p(t)$ then,

every solution to $x'(t) = A(t)x(t) + b(t)$ looks like

$$x(t) = \underbrace{c_1 x_1(t) + \cdots + c_n x_n(t)}_{\text{general solution to the homogenous system}} + \underbrace{x_p(t)}_{\text{Particular solution}}$$

The cases that matter is when $A(t)$ is constant w/ real entries

Consider $x(t) = Ax(t) \rightarrow$ homogenous
With an ↑ constant
eigenvalue of λ w/ eigenvector V. Then,
$x(t) = e^{\lambda t}V$ is a solution to $x'(t) = Ax(t)$

Similarly, if $v_1, \dots, v_n$ are linearly independent eigenvectors for A w/ eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly not distinct)
The general solution to $x'(t) = Ax(t)$ is given by

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t}$$

Example: Find the general solution to

$x_1'(t) = x_1 + 2x_2$
$x_2'(t) = 2x_1 + -2x_2$
$\rightarrow x'(t) = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} x(t)$
↑ A

eigenvalues: $\lambda_1 = -3$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$\lambda_2 = 2$ $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

general solution:

$$x(t) = e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$x_1(t) = e^{-3t} - 2e^{2t}$$
$$x_2(t) = 2e^{-3t} + e^{2t}$$

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Example #2: Find the general solution to

$X' = AX$

$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ 1. Find eigenvalues & eigenvectors

$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4$

eigenvalues corresponding

to $p(\lambda) = 0 \rightarrow \lambda_1 = 2 \longrightarrow (A - 2I)v = 0$

$\lambda_2 = -2$

$\lambda_1 = -2 \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

$\hookrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = -2y$

$\begin{bmatrix} -2 & 4 & 0 \\ 1 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $X = 2y$

$V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow$ genreal solution:

$X(t) = C_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$X_1 = 2C_1 e^{2t} - 2C_2 e^{-2t}$

$X_2 = C_1 e^{2t} + C_2 e^{-2t}$

Example #3: Find the general solution to $X' = AX$ where

$A = \begin{bmatrix} 0 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow P(\lambda) = \begin{vmatrix} -\lambda & -3 & 1 \\ -2 & -1-\lambda & 1 \\ 0 & 0 & \lambda \end{vmatrix} = (2-\lambda)[(-\lambda)((-1-\lambda) - 6)]$

$= 2 - \lambda(\lambda^2 + \lambda - 6)$

$= (2-\lambda)(\lambda+3)(\lambda-2)$

$= (\lambda+3)(\lambda-2)^2(-1)$

$\lambda_1 = -3 \quad \lambda_2 = 2$

↕ multiplicity x2

$\begin{bmatrix} 3 & -3 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow X = y \quad Z = 0$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \begin{bmatrix} -2 & -3 & 1 & 0 \\ -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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~~MATH 1068 Final Topic List~~ Complex systems of equations

Solve the IVP:

$x' = Ax$ → Where $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1-2i \end{bmatrix}$

$X(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $P(\lambda) = \begin{vmatrix} -0\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 4$ $\lambda_1 = 2i$ & $\lambda_2 = -2i$

$\lambda_1 = 2i$ $\begin{bmatrix} -2i & 2 & | & 0 \\ -2 & -2i & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} i & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ $y = ix$ $V_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ & $V_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$e^{\lambda t} V_1$ is still a solution but...

$e^{2it} = \cos(2t) + i\sin(2t)$ ← euler's Identity

$(\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos(2t) + i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{bmatrix}$

$\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + i\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$ → real, linearly Independent ~~solutions~~ solutions!

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Find the General Solution to

$X' = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix} X$ $\quad P(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 10$

$\lambda = 3 \pm i$ ← eigenvalues

↳ note: Since the eigenvalues are complex we need only solve one of these complex conjugates

$\lambda = 3+i \rightarrow -(1+i)v_1 - v_2 = 0 \rightarrow$ Thus the complex eigenvectors are

$2v_1 + (1-i)v_2 = 0 \quad V = \begin{bmatrix} 1 \\ -1-i \end{bmatrix} \rightarrow$ Using this vector to find the general sol: ←

General Solution:

$$X(t) = e^{3t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = e^{3t} \begin{bmatrix} \cos(t) + i\sin(t) \\ (-1-i)(\cos(t) + i\sin(t)) \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{bmatrix} + e^{3t} \begin{bmatrix} \sin(t) \\ -\sin(t) - \cos(t) \end{bmatrix}$$

Note: The real & Imaginary parts of this complex solution yield the following two real linearly Independent solution →

$$X_1(t) = e^{3t} \begin{bmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{bmatrix} \quad \& \quad X_2(t) = e^{3t} \begin{bmatrix} \sin(t) \\ -\sin(t) - \cos(t) \end{bmatrix}$$

Hence the <u>Real Valued</u> General Solution:

$$X(t) = C_1 e^{3t} \begin{bmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin(t) \\ -\sin(t) - \cos(t) \end{bmatrix}$$

Theorem: Suppose $\lambda = a + ib$ is a complex eigenvalue of a matrix A with corresponding linearly indepent eigenvectors $V_1, V_2, \ldots, V_k$ where $V_j = r_j + is_j$. Then k complex solutions to $X' = Ax$ are $u_j(t) = e^{\lambda t} V_j \quad j = 1, 2, \ldots, k$

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MATH 165 2018 final: Annotated Solutions:

1a (11 Points) Find a solution to the differential equation:

$x\ln(x)y' - y^2 = 1$ where $y(e) = 1$

↳ Seperable Differential Equation!

$x\ln(x)y' = 1 + y^2$

$\frac{1}{1+y^2}y' = \frac{1}{x\ln(x)} \rightarrow \int \frac{1}{1+y^2}dy = \int \frac{1}{x\ln(x)}dx$

This is $\frac{d}{dx}(\arctan(x))$ ←

$\Rightarrow \arctan(y) = \ln(\ln(x)) + C$

$y = \tan(\ln(\ln(x)) + C$ ← General Solution!

$y(e) = \tan(\ln(\ln(x) + C))$

$1 = \tan(C) \rightarrow C = \pi/4$

Answer: $y(x) = \tan(\ln(\ln(x) + \pi/4)$

1b (11 Points) Find the general solution to the differential equations

$xy' - 3y = x^8 \rightarrow$ Integrating Factor

↳ $y' - \frac{3}{x}y = x^7$ → Let $I(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$

$x^{-3}y = \int x^{-3}x^7 dx \rightarrow \int x^4 dx =$ General solution:

$\frac{x^5}{5} + C = y(x)$

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2. (15 Points) Find the inverse of the matrix:

$$A=\begin{bmatrix}1&2&1\\0&1&3\\-2&-2&5\end{bmatrix} \rightarrow \text{Gausse-Jordan elimination}$$

$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&3&0&1&0\\-2&-2&5&0&0&1\end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&3&0&1&0\\0&2&7&2&0&1\end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{R_1-2R_2\&} \left[\begin{array}{ccc|ccc}1&0&-5&1&-2&0\\0&1&3&0&1&0\\0&0&1&2&-2&1\end{array}\right] \xrightarrow[R_2-3R_3]{R_1+5R_3\&} \left[\begin{array}{ccc|ccc}1&0&0&11&-12&5\\0&1&0&-6&7&-3\\0&0&1&-2&-2&1\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}11&-12&5\\-6&7&-3\\-2&-2&1\end{bmatrix}$$

4a. Find the determinant of (8 points)

$$M=\begin{bmatrix}3&2&0\\5&2&1\\-1&7&0\end{bmatrix} \rightarrow \text{Cofactor Expansion:}$$

$$3\begin{vmatrix}2&1\\7&0\end{vmatrix} - 2\begin{vmatrix}5&1\\-1&0\end{vmatrix} + 0$$

$$3(-7)-(-2)(-1)$$

$$-21-2 = -23$$

4b. (10 points) Suppose A is a 4x4 matrix & $\det(A)=2$ & B is obtained from A by adding 5 times row 2 to row 3

1. $\det(3A) = 3^4(2)$
2. $\det(A^T) = \det(A) = 2$
3. $\det(A^{-1}) = \frac{1}{\det(A)} = 1/2$
4. $\det(A^3) = (\det(A))^3 = 8$
5. $\det(B) = \det(A) = 2$

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5. (15 Points) Determine which of the following are ~~subsets~~ Subspaces of $\mathbb{P}_3$?

a. $S_1 = \{p(t) \in \mathbb{P}_3 \mid p'(t) + 2p(t) + 7 = 0 \text{ for all } t\}$

↳ NO! does not contain the zero vector

b. $S_2 = \{p(t) \in \mathbb{P}_3 \mid p(0) = 1\}$

↳ NO! does not include the zero-vector

c. $S_3 = \{p(t) \in \mathbb{P}_3 \mid p(-t) = p(t) \text{ for all } t\}$

↳ $at^3 + bt^2 + ct + d = -at^3 + bt^2 - ct + d$

$2a = 0$, $2c = 0$ $\Big\} a = c = 0$ Thus a basis is: $\{t^2, 1\}$

↳ Yes! With dimension = 2

d. $S_4 = \{p(t) \in \mathbb{P}_3 \mid p'(3) = p(1)\}$

$3a(9) + 2b(3) + c = a + b + c + d$

$26a + 5b - d = 0$

$d = 26a + 5b$ → Yes! with dimension = 3

6. (15 points)

$$A = \begin{bmatrix} 3 & -6 & 1 & 3 & 0 \\ 2 & -4 & 1 & -1 & 0 \\ 3 & -6 & 0 & 12 & 1 \end{bmatrix} \text{ in RREF } \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a. What is the rank of A?

→ number of leading 1s in the RREF → 3

b. What is the nullity of A?

→ number of colums − rank(A) → 2

c. What are the basis vectors for the col. space of A

→ the colums in the non RREF that correspond to the cols of the RREF that contains a leading one: $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

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$$A=\begin{bmatrix}3&-6&1&3&0\\2&-4&1&-1&0\\3&-6&0&12&1\end{bmatrix}\xrightarrow{\text{InRREF}}\begin{bmatrix}1&-2&0&4&0\\0&0&1&-9&0\\0&0&0&0&1\end{bmatrix}$$

d. List the basis vectors for

$x_1 = 2x_2 - 4x_4$ $N = (2x_2 - 4x_4, x_2, 9x_4, x_4, 0)$

$x_3 = 9x_4$

$x_5 = 0$

$$\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\9\\1\\0\end{bmatrix}$$

Part B:

1a. Find the general solution to the associated homogenous

$(D^2+1)^2(D+2)y = X$

roots: -2, i (multiplicity 2)

$y(x)_c = C_1\cos(x) + C_2\sin(x) + C_3x\sin(x) + C_4x\cos(x) + C_5e^{-2x}$

1b. Find a particular solution

$y(x)_p = A + Bx$

$= (D^2+1)^2(D+2)(Ax+B)$

$= (D^2+1)^2(B+2A+2Bx)$

$= (D^2+I)(B+2A+2Bx) = B+2A+2Bx = X$

$2B = 1$

$B+2A = 0 \rightarrow B = 1/2$

$1/2 = -2A \rightarrow A = -1/4$

$y(x)_p = \frac{-1}{4} + \frac{1}{2}x$

1C. Determine the General Nonhomogenous DE

$y(x) = C_1\cos(x) + C_2\sin(x) + C_3x\cos(x) + C_4x\sin(x) + C_5e^{-2x} - \frac{1}{4} + \frac{1}{2}x$

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2. $A=\begin{bmatrix}2&0&0\\0&1&0\\2&-1&1\end{bmatrix}$ 2a. (4 Points) determine the eigenvalues of A

$\begin{vmatrix}2-\lambda&0&0\\0&1-\lambda&0\\2&-1&1-\lambda\end{vmatrix} = (2-\lambda)\begin{vmatrix}1-\lambda&0\\-1&1-\lambda\end{vmatrix} \rightarrow (1-\lambda)^2(2-\lambda)$

$P(\lambda)=(2-\lambda)(1-\lambda)^2$

$\lambda_1=2,\ \lambda=1$ (w/ multiplicity = 2)

2b. Determine the Eigenspace of each Eigenvalue

$\lambda_1=2\ \left[\begin{array}{ccc|c}0&0&0&0\\0&-1&0&0\\2&-1&-1&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}2&-1&-1&0\\0&-1&0&0\\0&0&0&0\end{array}\right]$ ~~x_3 is free~~ $x_2=0$ $2x_1-x_3=0$ $x_3=2x_1$ $\begin{bmatrix}1\\0\\2\end{bmatrix}$

$\lambda_2=1\ \left[\begin{array}{ccc|c}1&0&0&0\\0&0&0&0\\2&-1&0&0\end{array}\right]$ $x_1=0$ $2x_1-x_2=0$ $x_2=0$ x_3 is free $\begin{bmatrix}0\\0\\t\end{bmatrix}$ for all $t\in\mathbb{R}$

2c. Determine if A is defective (4 points)

→ Defective! The dimension of E_{λ_2} is 1, but the multiplicity of λ_2 is 2

3. (17 Points) Solve the initial value problem

$y''+2y'+5y=0$ Where $y(0)=1$ & $y'(0)=2$

↳ $(D^2+2D+5)y=0$ $\frac{-2\pm\sqrt{4-4(5)}}{2} \rightarrow \frac{-2\pm\sqrt{-16}}{2} = -2\pm 4i = -1\pm 2i$

roots: $(-1\pm 2i)$

general solution: $y_c = \cos(2x)+\sin(2x)+e^{-x}$

~~$c_1\cos(2x)+c_2\sin(2x)+c_3e^{-x}$~~

$=e^{-x}(c_1\cos(2x)+c_2\sin(2x))$

$y(x)=(\cos(2x)+\sin(2x))e^{-x}$

$1=c_1\cos(0)+c_2\sin(0)$

$2=-e^0(c_1\cos(0)+c_2\sin(0))+e^{-x}(-c_1\sin(0)+c_2\cos(0))$

$=-c_1\cos(0)+-c_2\sin(0)+-c_1\sin(0)+c_2\cos(0)$

$-1c_1(\cos(0)+\sin(0)-c_2(\sin(0)+\cos(0)=2$

$-c_1-c_2=2$ $c_1+c_2=2$

$c_1=-2-c_2$ $c_1=c_2=1$ ✓

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Find the general solution to:

$X' = Ax$ where $A = \begin{bmatrix} 0 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

1. Find the eigenvalues & vectors

$$P(\lambda) = \begin{vmatrix} -\lambda & -3 & 1 \\ -2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$= (2-\lambda)[(-\lambda)(-1-\lambda)-6]$

$= 2-\lambda(\lambda^2+\lambda-6)$

$= (2-\lambda)(\lambda+3)(\lambda-2)$

$= (\lambda+3)(-1)(\lambda-2)^2$

$\lambda_1 = -3$

$\lambda_2 = 2$ (w/ multiplicity 2)

$\lambda_1 = -3$

$$\begin{bmatrix} 3 & -3 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$X = y, \quad Z = 0 \longrightarrow V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$X_2 = 5$

$X_3 = t$

$X_1 = \frac{1}{2}t - \frac{3}{2}5$

$$V = \begin{bmatrix} \frac{1}{2}t - \frac{3}{2}5 \\ 5 \\ t \end{bmatrix}$$

$$= 5\begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

← both of these vectors correspond to $\lambda_2 = 2$

General Solution:

$$X(t) = C_1 e^{2t}\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_3 e^{-3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

TO Calculate the eigenvectors solve $(\lambda I - A)x = 0$ Where x is the eigenvector

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