Basic Properties of Sets

A **set** is an unordered collection of distinct objects; which are called *elements*. The set that does not contain any elements is appropriately refered to as the *empty* set, denoted $\emptyset$

Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$ two-set case

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$ n-set case

Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$ two-set case

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$ n-set case

If the intersection of any number of sets is the empty set then we call them *disjoint*

The Cartesian Product $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

The Power Set of A is $\mathcal{P}(A) = \{X : X \subseteq A\}$

The **Cardinality** of a set A denotes the number of elements in the set & is written as $|A|$

For a function $f : A \to B$ A is called the *domain* & B is called the *codomain* & the set $\{f(x) : x \in A\}$ is the *range* of f

- ↳ if $f(a) = f(b)$ implies $a = b$ then f is **injective** (or one-to-one)
- ↳ If for every $b \in B$ there is some $a \in A$ such that $f(a) = b$ the f is **surjective** (or onto)
- ↳ if f is both injective & surjective then f is **bijective**

Introduction to Numbers I

$\mathbb{N}$ The Natural Numbers

We can define the natural numbers from the **Zermelo-Fraenkel Set Theorhetic Axioms** with **The Axiom of Choice** (ZFC Axioms)

We begin by represent the number zero 0 as the empty set $\emptyset$. We then define a *successor function* $S(x)$ of a set x to be the set given by $S(x) = x \cup \{x\}$. Applying this function iteravily yields:

$$0 = \emptyset$$
$$1 = \{\emptyset\} = \{0\}$$
$$2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$$
$$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$$
$$\vdots$$

which can be continued for an arbitrarily amount of times to express any natural number. We define $\mathbb{N}_0$ to be the set of all of these numbers (represented as sets).

With this set of objects established, we next turn our attention to defining the *ordering* of $\mathbb{N}_0$

$n < m$ if $n \in m$ that is, n is less than m if n is an element in m.

The last step in this construction is to show that $\mathbb{N}$ satisfies closure under addition & multiplication but not, subtraction/inverses. This makes $(\mathbb{N}_0, +, \cdot, <)$ an **ordered semi-ring**.

To turn $(\mathbb{N}_0, +, \cdot, <)$ into an ordered ring we must define the negative numbers: $\mathbb{Z}$ which is the smallest ordered ring, containing $\mathbb{N}_0$

Introduction to Numbers II

$\mathbb{Q}$ The Rational Numbers

$\mathbb{Q}: \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$ The name rational comes from the word "ratio" due to the fraction that appears in their definition. The rational numbers have some fierce properties For any $p, q \in \mathbb{Q}$ then:

- ↳ There are infinitely many rational numbers between p & q
- ↳ $p + q \in \mathbb{Q}$ (closure under addition)
- ↳ $p - q \in \mathbb{Q}$ (closure under subtraction)
- ↳ $p \cdot q \in \mathbb{Q}$ (closure under multiplication)
- ↳ $p / q \in \mathbb{Q}$ (closure under division)

However, they are not closed under square-root this means that the rational numbers are not **algebraically closed**: There exists polynomials with rational coefficients whose roots are irrational

Theorem: The number $\sqrt{2}$ is irrational
Proof: Assume, for a contradiction that $\sqrt{2}$ is rational. Then, there must be two numbers p & q that are non-zero intergers such that $\sqrt{2} = \frac{p}{q}$ we assume that p & q have no common divisors, which yields the following $\sqrt{2}q = p$ & $2q^2 = p^2$ this implies that 2 is divisible by p^2 meaning 2 is divisible by p. So, we can write $p = 2k$ for some non-zero interger k, which yields

$$2q^2 = p^2 \rightarrow 2q^2 = (2k)^2 \rightarrow 2q^2 = 4k^2 \rightarrow q^2 = 2k^2$$

This implies that 2 is divisible by both q & q^2 which would contradict our assumption that p & q do not have any common factors. Thus $\sqrt{2}$ is irrational. **QED**

Fields & Ordered Fields

A **field** is a non-empty set $\mathbb{F}$ on which two binary operations are defined: addition (+) & multiplication (·), satisfying the following

Axiom 1: The Commutative Property

If $a, b \in \mathbb{F}$, then $a+b=b+a$ & $a \cdot b = b \cdot a$

Axiom 2: The Distrbutive Property

If $a, b, c \in \mathbb{F}$, then $a \cdot (b+c) = a \cdot b + a \cdot c$

Axiom 3: The Associative Property

If $a, b, c \in \mathbb{F}$, then $(a+b)+c = a+(b+c)$ &
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Axiom 4: The Identitys Propertys

There are special elements $0, 1 \in \mathbb{F}$, where
$a+0=a$ & $a \cdot 1 = a$ for all $a \in \mathbb{F}$

Axiom 5: The Inverse Propertys

For each $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a+(-a)=0$. If $a \neq 0$, then there is also an element $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$

Axiom 6: The Order Propertys

There is a non-empty subset $P \subseteq \mathbb{F}$, called the *positive elements* such that

a. If $a, b \in P$, then $a+b \in P$ & $a \cdot b \in P$ &
b. If $a \in \mathbb{F}$ & $a \neq 0$, then either $a \in P$ or $-a \in P$, but never both $a, -a \in P$

If a field satisfies the 6th axiom then we call it an **ordered field**

Metric Properties of Ordered Fields

Definition of Inequality:

If $\mathbb{F}$ is an ordered field & $a, b \in \mathbb{F}$, then we say "$a<b$" if $b-a \in P$. Likewise, we say "$a>b$" if $a-b \in P$. ($\leq$) & ($\geq$) are defined as the preceding two definitions respectively, or $a-b=0$ implying $a=b$

Definition of Absolute Value:

If $\mathbb{F}$ is an ordered field, the absolute value function $|\cdot|: \mathbb{F} \to \mathbb{F}$ to be $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

The properties of the absolute value are as follows, where a & b are elements in an ordered field:

a. $|a| \geq 0$, with equality if & only if $a=0$

b. $|a| = |-a|$

c. $-|a| \leq a \leq |a|$

d. $|a \cdot b| = |a| \cdot |b|$

e. $1/|a| = |1/a|$ if $a \neq 0$

f. $|a/b| = |a|/|b|$ if $b \neq 0$

The proofs for these properties are easy & I do not want to write all that tbh.

★ g. $|a| \leq b$ if & only if $-b \leq a \leq b$

★ h. The Triangle Inequality: $|a+b| \leq |a| + |b|$

★ i. The Reverse Triangle Inequality:

$||a| - |b|| \leq |a-b|$

In the following pages we will prove that the properties marked with ★ are true.

Absolute Value & Inequality

Proposition: $|a| \leq b$ if & only if $-b \leq a \leq b$; where $\mathbb{F}$ is an ordered field & $a, b \in \mathbb{F}$

Proof: Begin by assuming that $|a| \leq b$. Multiplying both sides by -1 tells us that $-|a| \geq -b$. (by property c.) we know that $-|a| \leq a \leq |a|$ Putting these facts together: $-b \leq -|a| \leq a \leq |a| \leq b$ which implies that $-b \leq a \leq b$.

To prove that the reverse is also true, assume that $-b \leq a \leq b$. Then, one of the following must be true

Case I: $a \geq 0$

If $a \geq 0$ then $|a| = a$ & thus since $a \leq b$ this implies that $|a| \leq b$ as desired

Case II: $a < 0$

If $a < 0$, then $-a = |a|$ & thus since $-b \leq a$ this implies that $-b \leq -|a|$ multiplying this by -1 yields $|a| \leq b$ as desired

Thus this implies that in either case, $|a| \leq b$ is true

QED

The Triangle Inequality

Theorem: If $\mathbb{F}$ is an ordered field & $x, y \in \mathbb{F}$. Then $|x+y| \leq |x| + |y|$

Proof: We want to show that $|x+y| \leq |x|+|y|$ which, according to the proposition on the previous page, is true if & only if:

$$-(|x|+|y|) \leq x+y \leq |x|+|y|$$
$$-|x|-|y| \leq x+y \leq |x|+|y|$$

& these sums can be split into two parts. According to property c. of the absolute value we have:

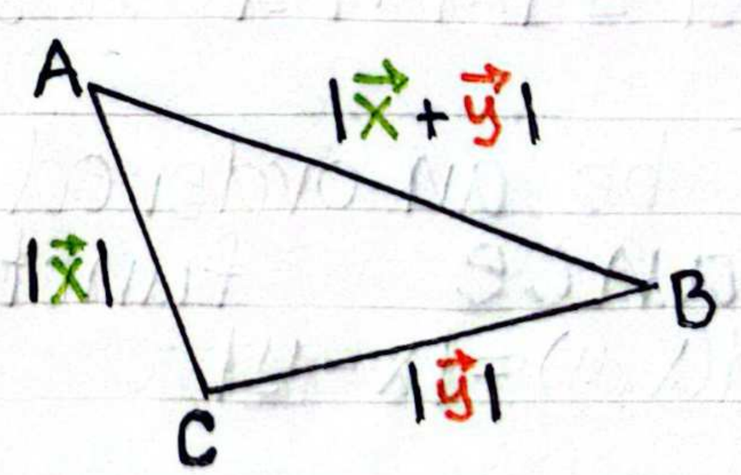
$$-|x| \leq x \leq |x| \quad \& \quad -|y| \leq y \leq |y|$$

Adding these two together yields the following $-|x|-|y| \leq x+y \leq |x|+|y|$ which, by the associative property/the distributive property is equal to $-(|x|+|y|) \leq x+y \leq |x|+|y|$ & so by the proposition on the previous page, we have:

$$|x+y| \leq |x|+|y|$$

QED

We will find ourselves coming back to the triangle inequality as a tool in real analysis again & again. A vector $\vec{x}$ has length $|\vec{x}|$ so if we consider a pair of vectors $\vec{x}$ & $\vec{y}$ as well as the vector $\vec{x}+\vec{y}$, then the lengths can be pictured as a triangle.



The shortest path between two points on a plane is a line. Therefore, the distance from A to C ($|\vec{x}+\vec{y}|$) is the shortest path, it is a longer route to go from A to B to C. that is what the triangle inequality states.

The Reverse Triangle Inequality

Theorem: Assuming that $\mathbb{F}$ is an ordered field, & that $x, y \in \mathbb{F}$ then $||x|-|y|| \leq |x-y|$

Proof: We want to show that $-|x-y| \leq |x|-|y| \leq |x-y|$
We will first show that $|x|-|y| \leq |x-y|$.
Let $a = x-y$ & $b = y$. Then, by the triangle inequality $|a+b| \leq |a|+|b|$ That is,

$$|(x-y)+y| \leq |x-y|+|y|$$
$$|x| \leq |x-y|+|y|$$
$$|x|-|y| \leq |x-y|$$

Next, we apply a similar approach to show that $-|x-y| \leq |x|-|y|$ Let $c = y-x$ & $d = x$ Then, by The triangle inequality we have $|c+d| \leq |c|+|d|$ that is,

$$|(y-x)+x| \leq |y-x|+|x|$$
$$|y| \leq |y-x|+|x|$$
$$-|y-x| \leq |x|-|y|$$

This implies that:
$-|x-y| \leq |x|-|y|$

Thus, we have shown that
$-|x-y| \leq |x|-|y| \leq |x-y|$
meaning that
$||x|-|y|| \leq |x-y|$

QED

Corollary: If $\mathbb{F}$ is an ordered field & $x, y \in \mathbb{F}$ then
$|x-y| \leq |x|+|y|$ & $|x+y| \geq ||x|-|y||$

Let $\mathbb{F}$ be an ordered field, we define the **distance** function $d: \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ by
$d(x,y) = |x-y|$

The Completeness Axiom Preliminaries

We need but a single axiom more before we are ready to construct the real numbers. However, this axiom requires some prior definitions.

Let S be an ordered field & $A \subseteq S$ be non-empty

The set A is **bounded above** if there exists some $b \in S$ such that $x \leq b$ for all $x \in A$; in this case b is called an **upper bound** of A.

The **least upper bound** of A - if existent - is some $b_0 \in S$ such that:

1. b_0 is an upper bound of A
2. $b_0 \leq b$ for all upper bounds of A, b

b_0 is called the **supremum** of A & is denoted by **sup**(A)

The set A is **bounded below** if there exists some $b \in S$ such that $x \geq b$ for all $x \in A$; in this case b is called a **lower bound** of A

The **greatest lower bound** of A - if existent - is some $b_0 \in S$ such that

1. b_0 is a lower bound of A
2. $b_0 \geq b$ for all lower bounds of A b

b_0 is called the **infimum** of A & is denoted **inf**(A)

If a set is both bounded above & bounded below then we simply call it **bounded**

The Completeness Axiom

Let S be an ordered field. Then S has the **least upper bound property**: If given any non-empty $A \subseteq S$ where A is bounded above, then A has a least upper bound. In other words, $\sup(A) \in S$ for every set A. Such a set S is thus called complete.

Key Idea: The real numbers are what you get when you start with the rational numbers $\mathbb{Q}$ & then add in just enough numbers so that it satisfies the least upper bound property, called completeness

By doing this we add in all the irrational numbers like $\sqrt{2}$ & π, whilst avoiding the inclusion of so-called "not real" numbers like $2+i$ or the ordinal number ω

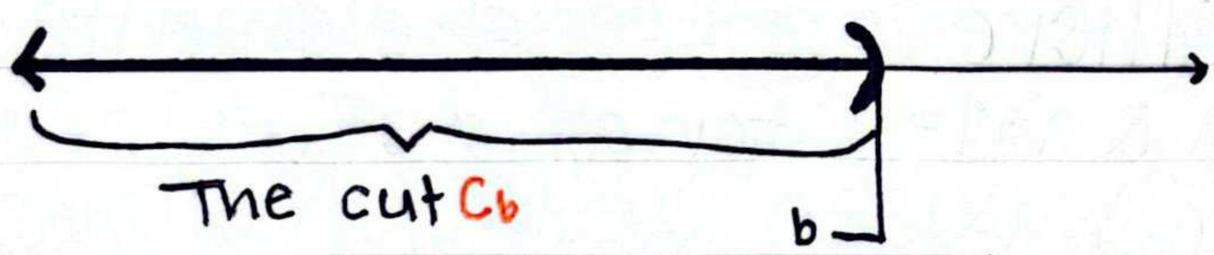
Theorem: Existence & Uniqueness of $\mathbb{R}$
There exists a unique complete orderd field; we call this field the *Real numbers* $\mathbb{R}$

Constructing the Real Numbers $\mathbb{R}$

We follow the so called *Dedkind cuts* method. Each real number is going to be a set of rationals. The real number x is going to be represented by the set of all rational numbers strictly less than x. These sets are refered to as *cuts*. A *cut* is the set $(\infty, b) \cap \mathbb{Q}$. That is, all rational numbers up to a certain point. Formally, it is defined as any set C_b satisfying the following three conditions

1. $C_b \subseteq \mathbb{Q}$, but $C_b \neq \emptyset$ & $C_b \neq \mathbb{Q}$
2. if $p \in C_b$ & $q \notin C_b$, then $p < q$
3. If $p \in C_b$, then there exists some $q \in C_b$ where $p < q$

Then $\mathbb{R}$ is defined as the set of all cuts



However, we still need to define the algebraic properties that we would expect the real numbers to have.

$C_a + C_b := \{p + q : p \in C_a, q \in C_b\}$
$C_a < C_b$ if & only if $C_a \subset C_b$

} Defining addition of these cuts goes smoothly

$C_a \cdot C_b := \{p \cdot q : p \in C_a, q \in C_b \text{ with } p, q \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}$

The definition of multiplication is slightly more annoying since negative times a negative must produce a positive. Thus, we know that $\mathbb{R}$ is an ordered field, the final step is to show that $\mathbb{R}$ is complete. That is, if $\mathcal{C}$ is a collection of cuts which is bounded above (meaning there exists D such that $C \leq D$ for all $C \in \mathcal{C}$) then there exists a least upper bound, meaning there is a cut M such that $M \leq D$ for all upper bounds D

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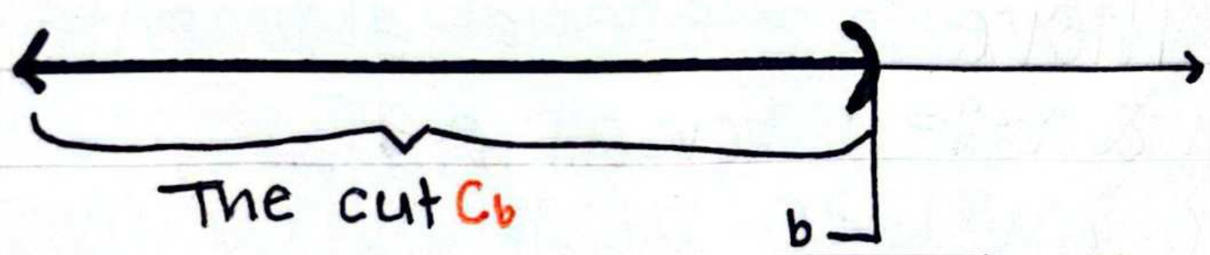


However, we still need to define the algebraic properties that we would expect the real numbers to have.

$$\left.\begin{array}{l} C_a + C_b := \{p+q : p \in C_a,\ q \in C_b\} \\ C_a < C_b \text{ if \& only if } C_a \subset C_b \end{array}\right\} \text{Defining addition of these cuts goes smoothly}$$

$$C_a \cdot C_b := \{p \cdot q : p \in C_a,\ q \in C_b \text{ with } p, q \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}$$

The definition of multiplication is slightly more annoying since negative times a negative must produce a positive. Thus, we know that $\mathbb{R}$ is an ordered field, the final step is to show that $\mathbb{R}$ is complete. That is, if $\mathcal{C}$ is a collection of cuts which is bounded above (meaning there exists D such that $C \leq D$ for all $C \in \mathcal{C}$) then there exists a least upper bound, meaning there is a cut M such that $M \leq D$ for all upper bounds D

The Axioms of $\mathbb{R}$

The set $\mathbb{R}$ has two binary operations, addition (+) & multiplication (·), & is the smallest set that contains $\mathbb{Q}$ & satisfies all of the following Axioms

1. **Commutative Property:** If $a, b \in \mathbb{R}$ then $a+b=b+a$ & $a \cdot b = b \cdot a$

2. **Associative Property:** If $a, b, c \in \mathbb{R}$ then $(a+b)+c = a+(b+c)$ & $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

3. **Distributive Property:** If $a, b, c \in \mathbb{R}$ then $a \cdot (b+c) = ab + ac$

4. **Identity Property:** There are special elements $0, 1 \in \mathbb{R}$ where $a+0=a$ & $a \cdot 1 = a$ for all $a \in \mathbb{R}$

5. **Inverse Property:** For each $a \in \mathbb{R}$ there is an element $-a \in \mathbb{R}$ such that $a+(-a)=0$, if $a \neq 0$ then there is also an element $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$

6. **Order Axiom:** There is a non-empty subset $P \subseteq \mathbb{R}$ such that: (Positive elements)
   1. If $a, b \in P$, then $a+b \in P$ & $a \cdot b \in P$
   2. If $a \in \mathbb{R}$ & $a \neq 0$, then either $a \in P$ or $-a \in P$, but not both

7. **Completeness Axiom:** Given an non-empty subset $A \subseteq \mathbb{R}$ where A is bounded above, A has a least upper bound. In other words, $\sup(A) \in \mathbb{R}$ for every such A.

Supremums & Infimums

Theorem: Uniqueness of Supremums & Infimums
If the supremum or infimum of $A \subseteq \mathbb{R}$ exists, then it is unique.

Proof: Assume for a contradiction that α & β are distinct least upper bounds of A, meaning $\alpha \neq \beta$
On one hand, since α is a least upper bound & β is an upper bound, we must have $\alpha \leq \beta$. On the other hand, since β is a least upper bound & α is an upper bound, we must have $\beta \leq \alpha$.
These two facts imply that $\alpha = \beta$ which contradicts our assumption that α & β are distinct showing the supremum to be unique

Similarly, now assume that α & β are distinct greatest lower bounds of A, meaning $\alpha \neq \beta$
On one hand, since α is a greatest lower bound & β is a lower bound we must have $\alpha \geq \beta$. On the other hand since β is a greatest lower bound & α is a lower bound, we must have $\beta \geq \alpha$
These two facts imply that $\alpha = \beta$ which contradicts our assumption that α & β are distinct showing that the infimum is unique.

Thus, if the supremum or infimum of $A \subseteq \mathbb{R}$ exists, then it is unique

QED

Existence of the Square-Root in $\mathbb{R}$

Theorem: If $a \in \mathbb{R}$ & $a > 0$, then $\sqrt{a} \in \mathbb{R}$

Proof: Let $a > 0$ be a real number. Define the set $S := \{x \in \mathbb{R} \mid x \geq 0 \text{ and } x^2 < a\}$. Since $a > 0$, we can infer that $x = \frac{a}{a+1} > 0 \Rightarrow x^2 = \left(\frac{a}{a+1}\right)^2 < a$ for all $a > 0$. Thus, $x \in S$, meaning $S \neq \emptyset$

let $b = \max\{1, a\}$. For all $x > b$ we have:

$$x^2 > x > a \Rightarrow x^2 > a$$

So any such $x \notin S$, meaning S is bounded above by b. Since S is nonempty & bounded above, the completeness axiom of $\mathbb{R}$ states that the supremum of S, say $s = \sup(S)$, exists & is a real number.

For the sake of contradiction suppose that $s^2 < a$ & choose a small $\varepsilon > 0$ such that $(s+\varepsilon)^2 < a$. Since $s = \sup(S)$ this implies that $s + \varepsilon \in S$, which contradicts $s = \sup(S)$

For the sake of our next contradiction suppose that $s^2 > a$ & choose a small $\varepsilon > 0$ such that $(s-\varepsilon)^2 > a$, but then $s - \varepsilon$ is an upper bound that is less than s, contradicting the fact that s is the least upper bound of S.

Since neither $s^2 < a$ nor $s^2 > a$ is possible, it must imply that $s^2 = a$. Thus, the number $s = \sup(S) \in \mathbb{R}$ satisfies $s^2 = a$, so we define $\sqrt{a} := s$ which shows that $\sqrt{a} \in \mathbb{R}$

QED

Analytic Definition of Supremum

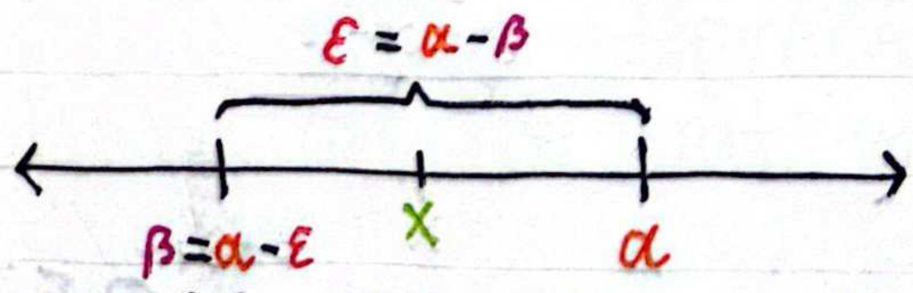
Theorem: Let $A \subseteq \mathbb{R}$. Then $\sup(A) = \alpha$ if & only if

1. α is an upper bound of A &
2. Given any $\varepsilon > 0$, $\alpha - \varepsilon$ is <u>**not**</u> an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \varepsilon$

Proof: We begin by assuming that $\sup(A) = \alpha$. By definition α is then a least upper bound, satisfying the first condition.

Next, we let $\varepsilon > 0$. Since $\alpha - \varepsilon < \alpha$, we know that $\alpha - \varepsilon$ is <u>**not**</u> an upper bound, because if it were that would contradict α being a **least** upper bound of A. And so, since $\alpha - \varepsilon$ is not an upper bound, there must be some $x \in A$ which is greater than $\alpha - \varepsilon$.

We now want to show that $\sup(A) = \alpha$, that is, we wish to prove that α is an upper bound of A, & for any other bound β, we have $\alpha \leq \beta$. The former of these statemetes is true by definition, to prove the latter we begin by assuming that β is some other upper bound of A, & assume for contradiction that $\beta < \alpha$. Since $\alpha - \beta > 0$ we will use that as our value of ε. Since $\varepsilon > 0$ there exists some $x \in A$ such that $x > \alpha - \varepsilon = \alpha - (\alpha - \beta) = \beta$



But this is a contradiction since we assumed that β was an upper bound of A, & yet we found an element $x \in A$ that is larger than β.

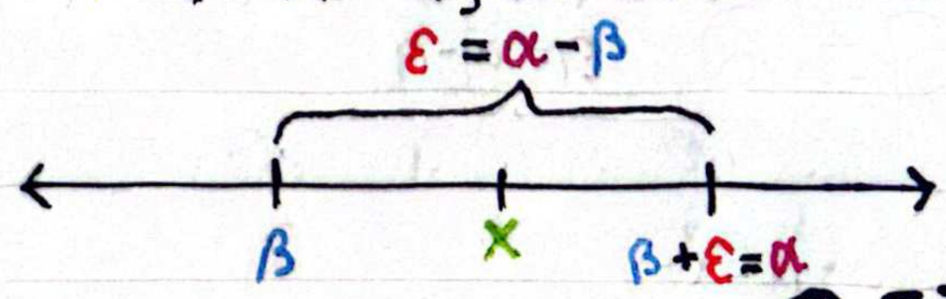
QED

Analutic Definition of Infimum

Theorem: Let $A \subseteq \mathbb{R}$. Then $\inf(A) = \beta$ if & only if

1. β is a lower bound of A &
2. Given any $\varepsilon > 0$, $\beta + \varepsilon$ is <u>**not**</u> a lower bound of A. That is, there is some $x \in A$ for which $x < \beta + \varepsilon$

Proof: We begin by assuming that $\inf(A) = \beta$. By definition β is then a greatest lower bound, satisfying the first condition.
Next, we let $\varepsilon > 0$. Since $\beta + \varepsilon > \beta$ we know that $\beta + \varepsilon$ is <u>**not**</u> a lower bound, there must be some $x \in A$ that is less than $\beta + \varepsilon$.
We now want to show that $\inf(A) = \beta$, that is, we wish to prove that β is a lower bound of (A), & that for any other bound α we have $\beta \geq \alpha$. The former of these statments is true by definition to prove the latter, we begin by assuming that α is some other lower bound of A, & assume for a contradiction that $\alpha > \beta$. Since $\alpha - \beta > 0$ we will use that as our value of ε. Since $\varepsilon > 0$ there exists some $x \in A$ such that $x < \beta + \varepsilon = \beta + (\alpha - \beta) = \alpha$ but this is a contradiction since we assumed that α was a lower bound of A, & yet we found an element $x \in A$ that is smaller than α



QED

The Archimedian Principle:

Theorem: If a & b are real numbers with $a>0$ then their exists a real number n such that $na>b$. In particular, for any $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $\frac{1}{n}<\varepsilon$

Proof: We want to show that: $-\varepsilon/2>-\varepsilon$ multiplying by two produces: $-\varepsilon>-2\varepsilon$ adding 2ε to both sides yields: $\varepsilon>0$ } reversing the order of this gives our proof.

Our goal is to show that $na>b$ for some $n\in\mathbb{N}$. By dividing over the a we find that this is equivalent to showing that $n>b/a$, for the sake of simplicity we will define $x=b/a$

Assume for a contradiction that there is no interger larger than x; that is assume x is an upper bound on $\mathbb{N}$. Then $\mathbb{N}$ is a subset of $\mathbb{R}$ that is bounded above & thus, by the completeness axiom we know that **sup**($\mathbb{N}$) exists. Call this supremum α. Since α is a *least* upper bound of $\mathbb{N}$, we know that $\alpha-1$ is <u>**not**</u> an upper bound... That is, there exists some interger $m>\alpha-1$. Adding 1 to each side yields $m+1>\alpha$. But! This is a contradiction If α is the supremum of $\mathbb{N}$, then it is an upper bound on $\mathbb{N}$, but we found $(m+1)\in\mathbb{N}$, which is larger than α. This shows the first proposition of the theorem is true, the second proposition follows immedeiately from this by letting $a=\varepsilon$, & $b=1$ then dividing over the n. **QED**

Supremum Example

Example (Proving the Supremum of a subset of $\mathbb{R}$).
Let $A = \{x \in \mathbb{R} : x < 0\}$. Show that $\sup(A) = 0$.

We need to show that i. 0 is an upper bound of A
ii. $0 - \varepsilon$ is <u>not</u> an upper bound of A

0 is clearly an upper bound as A is defined as the set of all real numbers that are less than zero thus showing that i. is true

To show that ii. is true, we let $\varepsilon > 0$. We need to show that there is some $x \in A$ such that $x > 0 - \varepsilon$; that is, $x > -\varepsilon$. $x = -\varepsilon/2$ satisfies this since we know that $\varepsilon > 0$

$-\varepsilon > -2\varepsilon$ (subtract 2ε from both sides)

$-\varepsilon/2 > -\varepsilon$ (Divide both sides by 2)

moreover, $\varepsilon > 0$ implies that $-\varepsilon < 0$, & hence $-\varepsilon/2 < 0$, which proves that $-\varepsilon/2 \in A$.

Thus, since $x = -\varepsilon/2$ has the properties that $x \in A$ & $x > 0 - \varepsilon$ this proves that zero meets the condition ii. that we would expect $\sup(A)$ to have.

As a result, we can conclude that 0 is the supremum of A.

QED

Archimedean Principle Example

Example (Using the Archimedean Principle)

Show that $\inf\left(\left\{\frac{1}{n} : n \in \mathbb{N}\right\}\right) = 0$

We need to show that i. 0 is a lower bound of A
ii. $0+\varepsilon$ is <u>not</u> a lower bound of A
where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ & $\varepsilon > 0$

i. follows from the fact that 1 & n are both positive for any $n \in \mathbb{N}$, so $1/n$ is also positive so $1/n > 0$ making 0 a lower bound of A

Finally, to show that 0 satisfies ii. let $\varepsilon > 0$. Then, by the Archimedean Principle there exists some $n \in \mathbb{N}$ such that $1/n < \varepsilon$. This element, $1/n$, is in A & less than $0+\varepsilon$, which means that $0+\varepsilon$ is <u>not</u> a lower bound of A.

Thus, since 0 satisfies both properties that we would expect from a greatest lower bound of A we can conclude that the following is true: $\inf\left(\left\{\frac{1}{n} : n \in \mathbb{N}\right\}\right) = 0$ QED

Suppose A & B are ordered fields, then we say that A is **Dense** in B if for any $x, y \in B$ there exists an $a \in A$ such that $x < a < y$

For example $\mathbb{Q}$ is dense in $\mathbb{Q}$ but $\mathbb{Z}$ is <u>not</u> dense in $\mathbb{Q}$

Preliminary Lemma & a proof

On the next page is an incredibly consequential proof, but we first must prove the following lemma used in its construction

Lemma: ($y-x>1 \Rightarrow \exists z \in \mathbb{Z}$ where $x<z<y$). Let $x,y,z \in \mathbb{R}$ If $y-x>1$, then there exists some $z \in \mathbb{Z}$ such that $x<y<z$

Proof: We begin by assuming that x & y are at least zero & consider the set $A=\{n \in \mathbb{N}_0 : n \leq x\}$ Since $x \geq 0$, this set is non-empty, & since it is a set of non-negative intergers which is bounded above by x, this set is finite. Let $\max(A)=M$ We want to show that $z:=M+1$ holds.

Since $M \in \mathbb{N}_0$, we know $z \in \mathbb{N}_0$. Furthermore, since z is larger than the largest element of A, z is not in A, implying that $x<z$.

$M \leq x$ implies that $M+1 \leq x+1 < y$, meaning that $x<z<y$.

The cases where x & y are not at least zero procede similarly. If both are negatives, then by considering $-x$ & $-y$ the above argument produces an integer z where $-y<z<-x$, showing that $-z$ satisfies the above lemma since $x<-z<y$.

& finally if x & y have different signs then $z=0$

QED

The Rationals are Dense in the Reals

Theorem: The rational numbers ($\mathbb{Q}$) are dense in the real numbers ($\mathbb{R}$)

Proof: Pick any $x, y \in \mathbb{R}$ where $x < y$. We want to show that there exists some $m, n \in \mathbb{Z}$ such that $x < \frac{m}{n} < y$ (meaning $m/n \in \mathbb{Q}$).

Firstly, if $x < 0 < y$ then our work is done since $0 \in \mathbb{Q}$. Furthermore, we may assume, then, that x & y are both positive since $0 < x < \frac{m}{n} < y$ implies that $-y < \frac{-m}{n} < -x < 0$ which means that if the theorem holds for the case in which x & y are both positive, then the theorem also holds in the case that x & y are both negative.

With the assumption that x & y are both positive & $x < y$ we know that $y - x > 0$. Thus, by the Archimedean property there exists some $n \in \mathbb{N}$ such that $n(y-x) > 1 \Rightarrow ny - nx > 1$.

Thus, by the lemma on the previous page, there is some integer m such that $nx < m < ny$ which can be equivalently be expressed as $x < \frac{m}{n} < y$ **QED**

Bijections & Cardinality

Recall: The **cardinality** of a set refers to the number of elements in the set.

The bijection principle: Two sets have the same size if & only if there is a *bijection* between them.

Example: The sets $\{1, 2, 3\}$ & $\{a, b, c\}$ have the same number of elements since their contents can be paired up. One possible bijection is as follows: $1 \leftrightarrow b$, $2 \leftrightarrow c$, $3 \leftrightarrow a$

In fact, this relationship between the cardinality of two sets & the functions that connect these two sets; let S & T be sets, then:

$|S| = |T|$ if & only if there is a *bijection* from S to T
$|S| \leq |T|$ if & only if there is an *injection* from S to T
$|S| \geq |T|$ if & only if there is a *surjection* from S to T

Sequins

A **sequence** of real numbers is merely a function $a: \mathbb{N} \to \mathbb{R}$. We will often denote sequences as a_n rather than $a(n)$

A sequence a_n is **bounded** if the range $\{a_n : n \in \mathbb{N}\}$ is bounded; i.e. if there exists some lower bound $L \in \mathbb{R}$ & some upper bound $U \in \mathbb{R}$ where $L \leq a_n \leq U$

Theorem: A sequence (a_n) is bounded if & only if there exists some $C \in \mathbb{R}$ for which $|a_n| \leq C$ for all n.

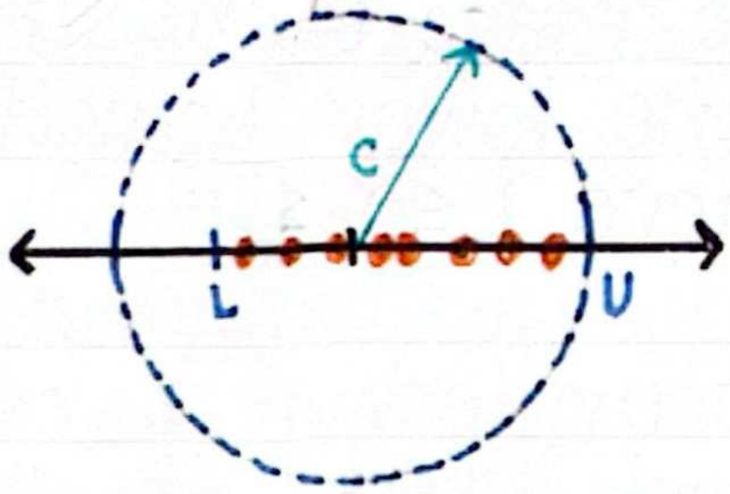
Proof: Recall that *boundedness* means that there exists a lower bound $L \in \mathbb{R}$ & an upper bound $U \in \mathbb{R}$ such that $L \leq a_n \leq U$.

Let's begin by proving the backwards implication. Assume that there exists a $C \in \mathbb{R}$ where $|a_n| \leq C$. Then we know that: $-C \leq a_n \leq C$. Thus, by setting $L = -C$ & $U = C$ we see that $L \leq a_n \leq U$

We next need to show that the forward implication also holds. If a_n is bounded, then there exists such an L & U. Let $C = \max\{|L|, U\}$. This implies that $C \geq U$ & $-C \leq -|L|$. Thus, for all n we have: $-C \leq -|L| \leq L \leq a_n \leq U \leq C$ so we see that $-C \leq a_n \leq C$ which is equivalent to saying that $|a_n| \leq C$ QED

Bounded Sequins II

One way to think about the forwards direction of the previous proof is to imagine drawing a circle with radius C centered at zero; such that the circle contains all the values of a_n.



Here $C = U$ since $|U| \geq |L|$

Examples:

Let $a_n = (-1)^n \frac{1}{n}$, then,

$$(a_n) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \frac{-1}{5}, \dots\right)$$

is bounded because $|a_n| \leq 1$ for all n.

Theorem: If (a_n) is a sequence in $\mathbb{R}$ & $a \in \mathbb{R}$, then the following statements are equivalent:

1. $a_n \to a$
2. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ $\forall n \geq N$
3. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $a - \varepsilon < a_n < a + \varepsilon$ $\forall n \geq N$
4. For every ε-neighborhood $V_\varepsilon(a)$ of a, $\exists N \in \mathbb{N}$ such that $a_n \in V_\varepsilon(a)$ $\forall n \geq N$

Convergant Sequins

A sequence (a_n) **converges** to $a \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists some N such that $|a_n - a| < \varepsilon$ for all $n > N$. When this happens, a is called the **limit** of (a_n).

The final condition that we are trying to reach is that $|a_n - a| < \varepsilon$ for an appropriate n. Note that this is the same as $-\varepsilon < a_n - a < \varepsilon$ which is equivalent to $a - \varepsilon < a_n < a + \varepsilon$. Which says that our sequence must live inside of this interval after some point N. We think of ε as being infinitely small, meaning that after some point N, we know that *every* term of the sequence will fall within an arbitrarily small interval In general, smaller values of ε typically demand larger values of N.

This definition of convergance demand that the sequence *gets* closer to a & *stays* closer to a as n increases. The consequence of this definition is that we can make the interval that a lies within infinitely small by increasing N, which provides us a powerful & rigorously defined tool for evaluating sequences.

This is why we want to make sure that our MCMC samplers converge; we want to ensure that the distribution of the samples is, & will stay sufficiently close to the posterior distribution.

Convergance Example

Show that the sequence $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ converges to zero.

Fix any $\varepsilon > 0$. By the Archimedean Property, there is an N for which $1/N < \varepsilon$. This N has the property that for all $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \varepsilon$ that is, for all $n > N$ $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$

& so by definition of sequence convergance we can conclude that $a_n \to 0$

However, using the Archimedean Principle will not work the majority of the time. We need a more general approach. The following procedure defines the process for showing that $a_n \to a$.

0. Start with $|a_n - a| < \varepsilon$ & unravel to solve for n. This tells you the value of N.
1. Let $\varepsilon > 0$
2. Let N be the final value of n obtained in step 0. & let $n > N$
3. Recalculate step 0. (without ε's), but at the end use N to show that $|a_n - a| < \varepsilon$

On the next page we will solve the same example as above, but using this new generalized procedure rather than the Archimedean Property.

Convergance Example (Round II)

Show that the sequence $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ converges to zero.

We want the following: $|a_n - a| < \varepsilon \to \frac{1}{n} < \varepsilon \to \frac{1}{\varepsilon} < n$

So, as long as we $\left|\frac{1}{n} - 0\right| < \varepsilon$

choose $N = 1/\varepsilon$, then for any $n > N$ we have $n > 1/\varepsilon$ which, by the above will imply that $1/n \to 0$

Fix $\varepsilon > 0$. Set $N = 1/\varepsilon$. Then, for any $n > N$ we will have $|a_n - a| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\varepsilon} = \varepsilon$

That is, $|a_n - a| < \varepsilon$. Thus, by definition $1/n \to 0$

Example 2: Let $a_n = 5 - \frac{1}{n^2}$ show that $a_n \to 5$ as $n \to \infty$

We want to show the following:

$|a_n - a| < \varepsilon \to \left|\left(5 - \frac{1}{n^2}\right) - 5\right| < \varepsilon \to \left|-\frac{1}{n^2}\right| < \varepsilon \to \frac{1}{n^2} < \varepsilon$

$\frac{1}{\varepsilon} < n^2 \to \frac{1}{\sqrt{\varepsilon}} < n$

Fix $\varepsilon > 0$. Set $N = 1/\sqrt{\varepsilon}$. Then for any $n > N$ $|a_n - a| = \left|\left(5 - \frac{1}{n^2}\right) - 5\right| = \frac{1}{n^2} < \frac{1}{N^2} = \frac{1}{1/(\sqrt{\varepsilon})^2} = \varepsilon$

That is, $|a_n - a| < \varepsilon$. Thus, by definition $5 - \frac{1}{n^2} \to 5$.

Another Convergence Example

Example 3: Let $a_n = \frac{3n+1}{n+2}$. Prove that $\lim_{n\to\infty}(a_n) = 3$

We want to show that $|a_n - a| < \varepsilon$

$$\left|\frac{3n+1}{n+2} - 3\right| < \varepsilon$$

$$\left|\frac{3n+1}{n+2} - \frac{3(n+2)}{n+2}\right| < \varepsilon$$

$$\left|\frac{3n+1-3n-6}{n+2}\right| < \varepsilon$$

$$\left|\frac{-5}{n+2}\right| < \varepsilon$$

$$\frac{5}{n+2} < \varepsilon \Rightarrow \frac{5}{\varepsilon} < n+2 \Rightarrow \frac{5}{\varepsilon} - 2 < n$$

Solution: Fix any $\varepsilon > 0$. Set $N = \frac{5}{\varepsilon} - 2$. Then for any $n > N$, $|a_n - a| = \left|\frac{3n+1}{n+2} - 3\right| = \left|\frac{3n+1}{n+2} - \frac{3n+6}{n+2}\right|$

$$= \frac{5}{n+2} < \frac{5}{N+2} = \frac{5}{(\frac{5}{\varepsilon}-2)+2}$$

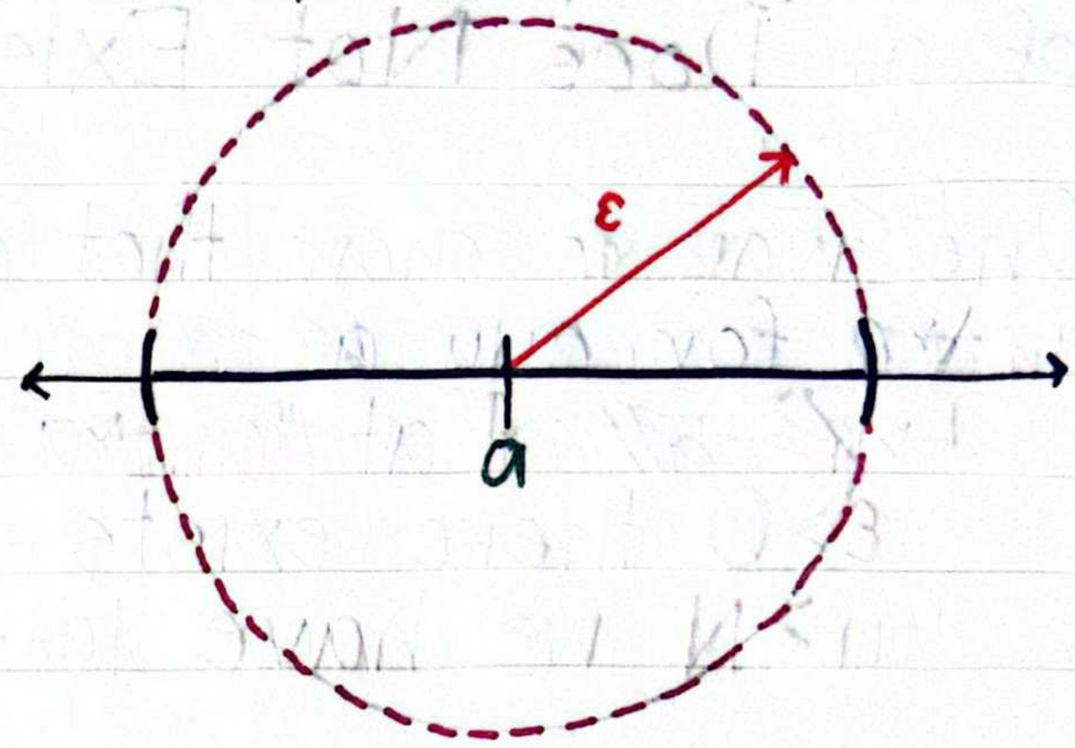
$$= \frac{5}{5/\varepsilon} = \varepsilon$$

That is, $|a_n - a| < \varepsilon$, so by definition we have shown that $\lim_{n\to\infty}\left(\frac{3n+1}{n+2}\right) = 3$

QED

Neighborhoods

Let $\varepsilon > 0$, the **ε-neighborhood** of a point a is the interval $(a-\varepsilon, a+\varepsilon)$. We can think of an ε-neighborhood as the interval centered at a with radius ε. We can rewrite our definition of convergance using this notion of a ε-neighborhood as follows: "A sequence (a_n) converges to $a \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists some N such that a_n is in the ε-neighborhood of a for all $n > N$"



of course, not all sequences converge. If a sequence (a_n) does not converge, then it **diverges**
Divergence comes in three flavors.

1. **Diverges to ∞** if for all $M > 0$ there exists some N such that $a_n > M$ for all $n > N$
2. **Diverges to $-\infty$** if for all $M < 0$ there exists some N such that $a_n < M$ for all $n > N$
3. Otherwise, the limit of a_n **does not exist!**

Example: Let $a_n = n^2$. Show that $\lim_{n\to\infty}(a_n) = \infty$

Fix any $M > 0$. Set $N = \sqrt{M}$. Then for any $n > N$

$$a_n = n^2 > N^2 = (\sqrt{M})^2 = M$$

Thus, we have shown that if $n > N$ then, $a_n > M$, therefore $\lim_{n\to\infty}(a_n) = \infty$

Divergance Proofs Procedure

How Do we Show Divergance if the Limit of a_n Does Not Exist?

One way to show that a_n diverges is to show that $a_n \not\to a$ for any a.

↳ *Recall:* that "$a_n \to a$" means that for every $\varepsilon > 0$ there exists some N where for all $n > N$ we have $|a_n - a| < \varepsilon$.

So, to show that $a_n \not\to a$, we need to show the negation of the above statement.

↳ There exists some $\varepsilon > 0$ where for all N there exists some $n > N$ such that $|a_n - a| \geq \varepsilon$

Note: In practice, this is usually done with a proof by contradiction: Assume that $a_n \to a$ & then demonstrate a specific value of ε where a contradiction follows.

We will explore two different proofs of the same divergent series on the next pages.

Divergance Example

Let $a_n = (-1)^n$. Prove that (a_n) diverges.

Scratch Work: $(a_n) = (-1, 1, -1, 1, -1, \ldots)$
It makes intuitive sense that this sequence diverges; it keeps alternating infinitely. It never gets & stays arbitravily close to a single value.

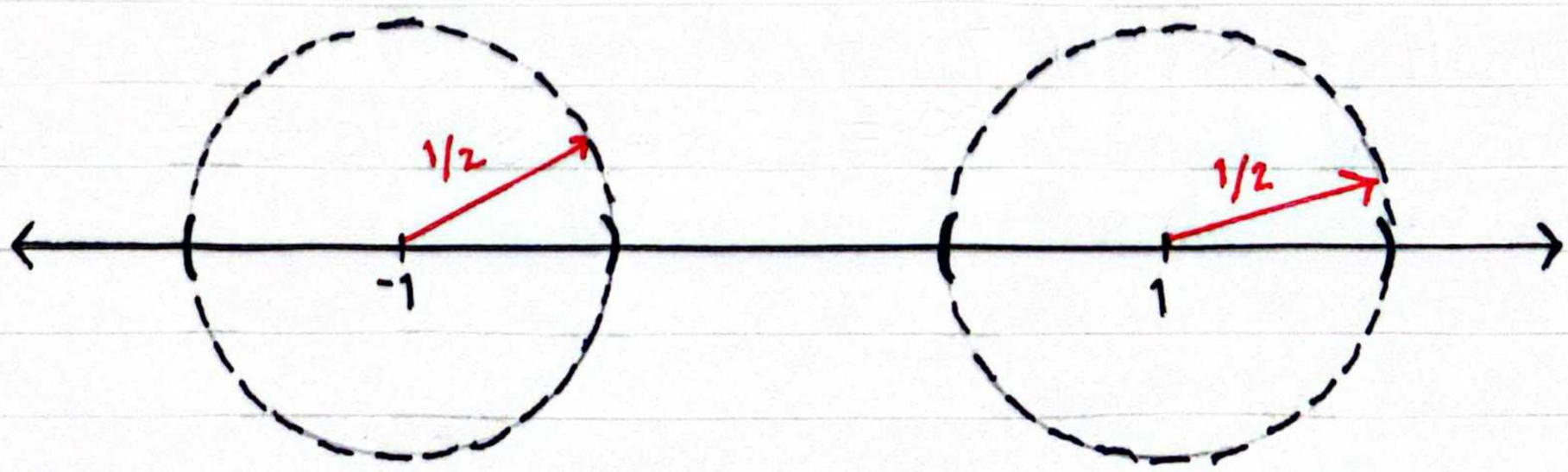
Solution #1: Assume for a contradiction that there is some a such that $a_n \to a$.
Let $\varepsilon = 1/2$. Since we assumed that $a_n \to a$, there must be some N for which, for all $n > N$, we have $|a_n - a| < 1/2$. That is, $|(-1)^n - a| < 1/2$ for all $n > N$. We procede with argument by cases:

Case 1: Even n
If $n > N$ & n is even then,
$|1 - a| < 1/2$
unwinding this:
$-1/2 < 1 - a < 1/2$
$-3/2 < -a < -1/2$
$1/2 < a < 3/2$

Case 2: Odd n
If $n > N$ & n is odd then,
$|-1 - a| < 1/2$
unwinding this:
$-1/2 < -1 - a < 1/2$
$1/2 < -a < 3/2$
$-3/2 < a < -1/2$

But this is a contradiction; clearly no a can be both inside both $(1/2, 3/2)$ & $(-3/2, -1/2)$ & so we must have $a_n \not\to a$. QED

We can visualize this as follows:



Divergance Example II

Let $a_n = (-1)^n$. Prove that (a_n) diverges.

Solution #2: The proof is by black magic. (the triangle inequality). Let $\varepsilon = 1/2$. The even case still yields $|1-a| < 1/2$, & the odd case still gives $|-1-a| < 1/2$; although in the odd case we will $|-1-a|$ as $|1+a|$. Then, by the triangle inequality

$$2 = |(1-a)+(1+a)| \leq |1-a| + |1+a|$$

But!

Each of the absolute values on the right are assumed to be less $1/2$. So,

$$2 = |(1-a)+(1+a)| \leq |1-a| + |1+a| < \frac{1}{2} + \frac{1}{2} = 1$$

So $2<1$ which is clearly a contradiction! QED

This idea of proving a contradiction by establishing disjoint neighboorhoods will be used to show an important property of sequence limits.

↳ They are unique! Each sequence can have at most a single limit.

Uniqueness of Limits

Theorem: A sequence cannot have more than one Limit.

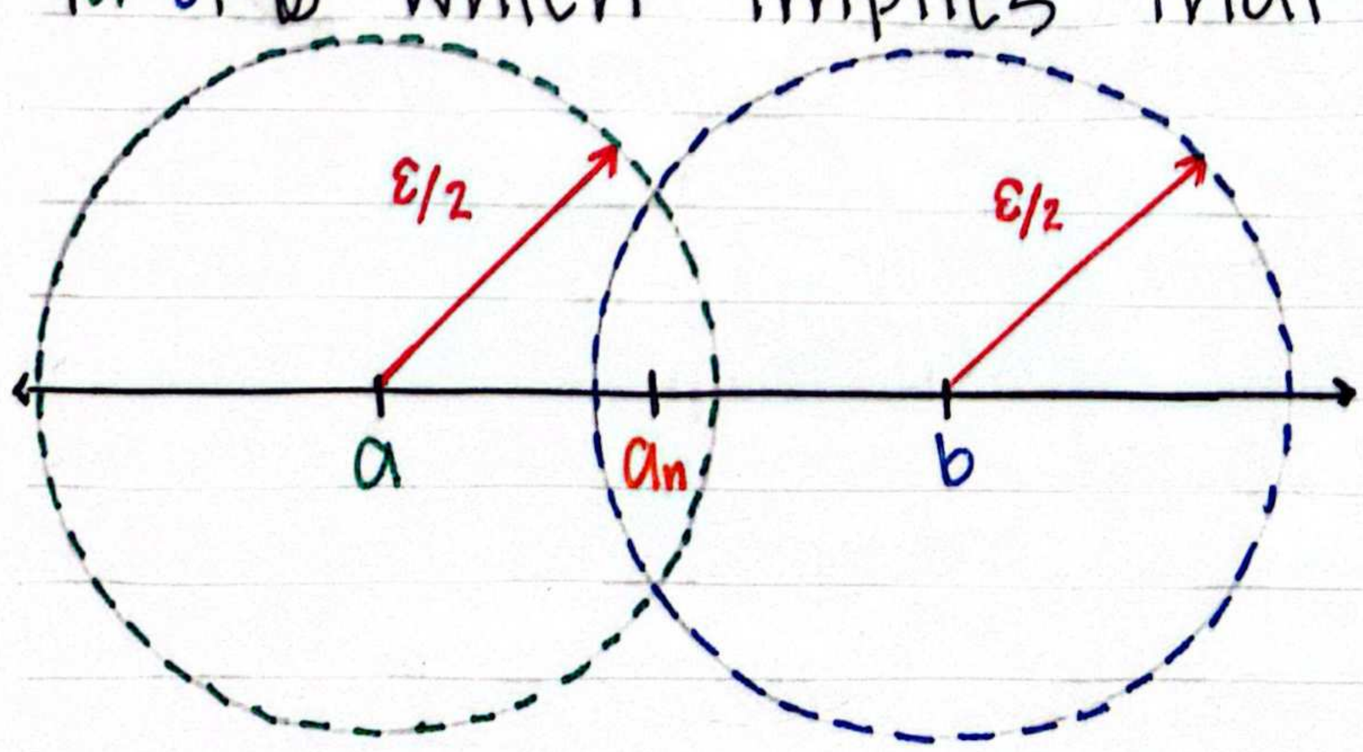
Proof: Suppose for a contradiction that $a_n \to a$ & $a_n \to b$ where $a \neq b$; moreover let's assume that $a < b$. Let $\varepsilon = \frac{b-a}{3} > 0$ Since $a_n \to a$ there exists some N_1 such that for $n > N_1$ we have $|a_n - a| < (b-a)/3$. Likewise, since $a_n \to b$ there exists some N_2 such that for $n > N_2$ we have $|a_n - b| < (b-a)/3$. Let $N = \max(N_1, N_2)$. Then for any $n > N$ (implying $n > N_2$ & $n > N_1$), we have both $|a_n - a| < (b-a)/3$ & $|a_n - b| < (b-a)/3$. That is, for such n we have: $-\frac{b-a}{3} < a_n - a < \frac{b-a}{3}$ & $-\frac{b-a}{3} < a_n - b < \frac{b-a}{3}$

$$a - \frac{b-a}{3} < a_n < a + \frac{b-a}{3} \quad \& \quad b - \frac{b-a}{3} < a_n < b + \frac{b-a}{3}$$

But since $a < b$, we must have $b + 2a < 2b + a$ but this implies that $a_n < \frac{b+2a}{3} < \frac{2b+a}{3} < a_n$

Which is clearly a contradiction. QED

There is a slightly shorter proof that relies on showing that $|a-b| < \varepsilon$ for all $\varepsilon > 0$, & therefore $|a-b| = 0$ which implies that $a = b$. Then we procede by



Using the triangle inequality we bound the distance from a to b by finding the distance from a to a_n plus the distance from a_n to b.

Uniqueness of Limits II

Theorem: A sequence cannot have more than one limit.

Proof: Let $\varepsilon > 0$. Since $\varepsilon/2 > 0$ & $a_n \to a$ there exists some N_1 such that for $n > N_1$ we have $|a_n - a| < \varepsilon/2$. Since $\varepsilon/2 > 0$ & $a_n \to b$ there exists some N_2 such that $n > N_2$ we have $|a_n - b| < \varepsilon/2$. Let $N = \max(N_1, N_2)$ & pick any $n > N$. Then,

$$
\begin{aligned}
|a-b| &= |a - a_n + a_n - b| \\
&\leq |a - a_n| + |a_n - b| && \text{[Triangle Inequality]} \\
&= |a_n - a| + |a_n - b| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}
$$

Since this holds for any $\varepsilon > 0$, we have shown that $|a-b| < \varepsilon$ for all $\varepsilon > 0$ which implies that $|a-b| = 0$, & hence $a = b$ meaning that a_n can only converge to a single point. **QED**

The idea of the above proof is to show that if a sequence has two valid limits, then they are actually equal to one another yielding a single unique limit.

Convergant $\Rightarrow$ Bounded

Theorem: If (a_n) is a convergant sequence then (a_n) is bounded

Proof: Since (a_n) is convergant, let a be the value that it converges to. By the definition of convergance (with $\varepsilon=1$), there is some N where $|a_n-a|<1$ for all $n>N$. That is, $a-1<a_n<a-1$ for all $n>N$. Let $U=\max(a_1,a_2,\dots,a_N,a+1)$ &
$L=\min(a_1,a_2,\dots,a_N,a-1)$
Note that if $n\leq N$, then $L\leq a_n\leq U$, since each a_n is included in the sets which we are taking the minimum & maximum of. And if $n>N$ then we already noted that $a-1<a_n<a+1$ which implies that $L\leq a-1<a_n<a+1\leq U$ & hence $L<a_n<U$. combining these cases we have that $L\leq a_n\leq U$ for all n. And thus, by definition (a_n) is bounded. **QED**

There is a second slightly shorter proof utilizing the absolute value of boundedness.

Proof: Since (a_n) is convergant, let a be the value it is converging to. By the definition of convergence (with $\varepsilon=0$) there is some N where $|a_n-a|<1$ for all $n>N$. That is $a-1<a_n<a+1$ for all $n>N$. Consider $C:=\max(|a_1|,|a_2|,\dots,|a_N|,|a+1|,|a-1|)$
Note that if $n\leq N$, then $a_n\leq|a_n|\leq C$. And if $n>N$, then $a-1<a_n<a+1$, which implies that:
$|a_n|\leq\max(|a-1|,|a+1|)\leq C$. Combining these cases, we have that $|a_n|\leq C$ for all n, which means that (a_n) is bounded. **QED**

Sequins Limit Laws

Theorem: Assume that (a_n) & (b_n) are convergent sequences of real numbers such that $a_n \to a$ & $b_n \to b$. Also assume that $c \in \mathbb{R}$. Then,

1. $(a_n + b_n) \to a + b$
2. $(a_n - b_n) \to a - b$
3. $(a_n \cdot b_n) \to a \cdot b$
4. $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ Provided that $b \neq 0$ & each $b_n \neq 0$
5. $(c \cdot a_n) \to c \cdot a$

Proof:

1. Given that $(a_n) \to a$ & $(b_n) \to b$, for all $\varepsilon > 0$ there exists $N_1, N_2 \in \mathbb{N}$ such that: $n \geq N_1 \Rightarrow |a_n - a| < \varepsilon/2$
   $n \geq N_2 \Rightarrow |b_n - b| < \varepsilon/2$

   Let $N = \max(N_1, N_2)$, then for $n \geq N$:
   $|(a_n + b_n) - (a - b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon$
   by the triangle inequality. QED

2. Given that $(a_n) \to a$ & $(b_n) \to b$, for all $\varepsilon > 0$ there exists $N_1, N_2 \in \mathbb{N}$ such that: $n \geq N_1 \Rightarrow |a_n - a| < \varepsilon/2$
   $n \geq N_2 \Rightarrow |b_n - b| < \varepsilon/2$

   Let $N = \max(N_1, N_2)$, then for $n \geq N$:
   $|(a_n - b_n) - (a - b)| = |(a_n - a) - (b_n - b)| \leq |a_n - a| - |b_n - b| < \varepsilon/2$
   by the triangle inequality. QED

5.

Sequins Limit Laws

Proof (Continued...):

3. Let $\varepsilon > 0$. Since (b_n) converges, we know that (b_n) is bounded; that is, there is some $C \in \mathbb{R}$ such that $|b_n| \leq C$ for all n. Let $\varepsilon_1 = \frac{\varepsilon}{2C+1}$ & $\varepsilon_2 = \frac{\varepsilon}{2|a|+1}$. Since $\varepsilon_1 > 0$ there exists some N_1 such that $|a_n - a| < \varepsilon_1$ for all $n > N_1$ & since $\varepsilon_2 > 0$ there exists some N_2 such that $|b_n - b| < \varepsilon_2$ for all $n > N_2$. Let $N = \max(N_1, N_2)$. Then for any $n > N$

$$
\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| \\
&\leq |a_n b_n - a b_n| + |a b_n - ab| \\
&= |a_n - a| \cdot |b_n| + |a| \cdot |b_n - b| \\
&< \varepsilon_1 \cdot C + |a| \cdot \varepsilon_2 \\
&= \frac{\varepsilon}{2C+1} \cdot C + |a| \cdot \frac{\varepsilon}{2|a|+1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

That is, for $n > N$ we have $|a_n b_n - ab| < \varepsilon$

Therefore $(a_n b_n) \to ab$. **QED**

4.

Example: Limit Laws

What is $\lim_{n\to\infty}\left(\frac{1}{2}\cdot\left(\frac{\frac{1}{n}+\frac{1}{n^2}+4}{5-\frac{1}{n^2}}\right)\cdot\left(\frac{3n+1}{n+2}+\frac{1}{\sqrt{n}}\right)\right)$

Solution: $\frac{1}{n}\to 0$ $\quad \frac{1}{n^2}\to 0$ $\quad \frac{3n+1}{n+2}\to 3$

$5\to 5$ $\quad 5-\frac{1}{n^2}\to 5$ $\quad \frac{1}{\sqrt{n}}\to 0$

So, by the limit laws, we get

$$\lim_{n\to\infty}\left(\frac{1}{2}\cdot\left(\frac{\frac{1}{n}+\frac{1}{n^2}+4}{5-\frac{1}{n^2}}\right)\cdot\left(\frac{3n+1}{n+2}+\frac{1}{\sqrt{n}}\right)\right)=\frac{1}{2}\cdot\left(\frac{0+0+4}{5}\right)\cdot(3+0)=\frac{6}{5}$$

Sequins Squeeze Theorem:

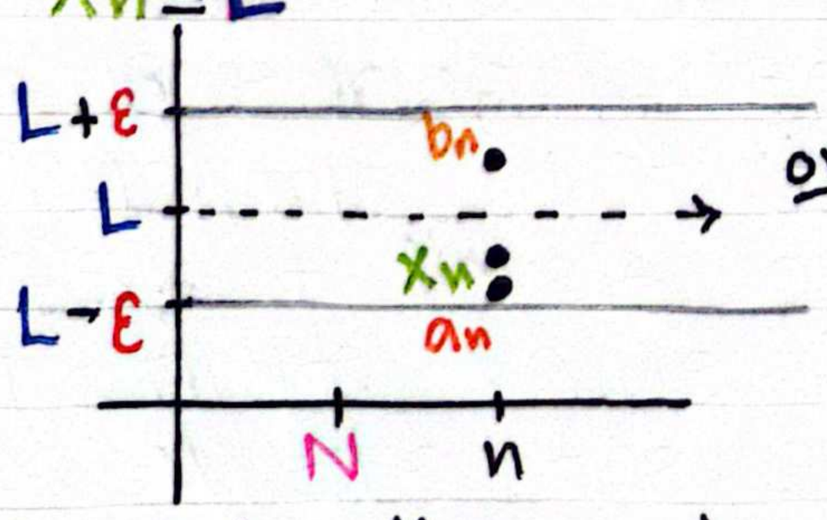
Theorem: Assume that $a_n \le x_n \le b_n$ for all n. Furthermore, assume that $a_n \to L$ & $b_n \to L$. Then, $x_n \to L$.

The proof of this theorem relies on the definition of convergance. We consider two cases:

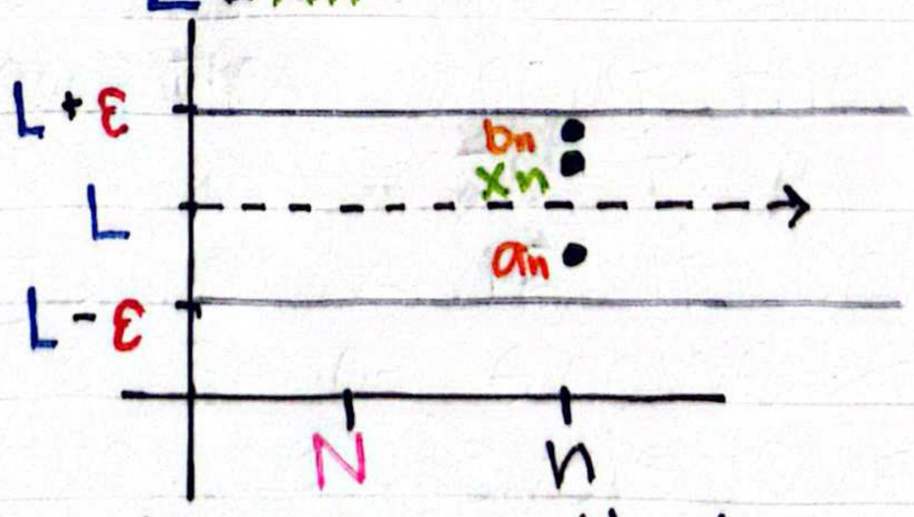
1. $x_n < L$
2. $x_n > L$

Suppose that we are past the N at which both a_n & b_n are within ε of L, & suppose that for some $n > N$ we have:

$x_n \le L$ $\qquad\qquad$ $L \le x_n$



or



Combined, these two plots show that $L-\varepsilon < x_n < L+\varepsilon$. The proof on the following page formallizes the intuition visualized by these plots.

Sequins Squeeze Theorem

Theorem: Assume that $a_n \le x_n \le b_n$ for all n. Furthermore, assume that $a_n \to L$ & $b_n \to L$. Then, $x_n \to L$.

Proof: Let $\varepsilon > 0$

Since $a_n \to L$ there exists some N_1 such that $n > N_1$ implies $|a_n - L| < \varepsilon$. That is, $-\varepsilon < a_n - L < \varepsilon$ or,

$$L - \varepsilon < a_n < L + \varepsilon \qquad (☕)$$

Since $b_n \to L$ there exists some N_2 such that $n > N_2$ implies $|b_n - L| < \varepsilon$. That is, $-\varepsilon < b_n - L < \varepsilon$ or,

$$L - \varepsilon < b_n < L + \varepsilon \qquad (🍵)$$

Let $N = \max(N_1, N_2)$, & let $n > N$ combining the inequality $a_n \le x_n \le b_n$ with the left half of (☕) & the right half of (🍵), we get:

$$L - \varepsilon < a_n \le x_n \le b_n < L + \varepsilon$$

$$L - \varepsilon < x_n < L + \varepsilon$$

$$-\varepsilon < x_n - L < \varepsilon \rightarrow |x_n - L| < \varepsilon$$

QED

Example: Prove that $\lim_{n\to\infty}\left[\frac{1}{n^{2.7}+\sqrt{n}+\pi}\right] = 0$

Solution: Since $0 \le \frac{1}{n^{2.7}+\sqrt{n}+\pi} \le \frac{1}{n}$ & $\lim_{n\to\infty}(0) = 0$ & $\lim_{n\to\infty}(1/n) = 0$ by the sequence squeeze theorem we have $\lim_{n\to\infty}\left[\frac{1}{n^{2.7}+\sqrt{n}+\pi}\right] = 0$

QED

Monotonicity

A sequence (a_n) is **monotone increasing** if $a_n \leq a_{n+1}$ for all n. Likewise, a sequence (a_n) is **monotone decreasing** if $a_n \geq a_{n+1}$ for all n. We call all functions that are either monotone increasing or monotone decreasing **monotone**.

↳ **Examples:** (n) is monotone increasing

$\left(\frac{1}{n}\right)$ is monotone decreasing

(1) is both monotone increasing & monotone decreasing

The Monotone Convergence Theorem:

Theorem: Suppose (a_n) is monotone. Then, (a_n) converges if & only if it is bounded. Moreover,

↳ If (a_n) is increasing, then either (a_n) diverges to ∞ or $\lim_{n\to\infty}(a_n) = \sup(\{a_n : n \in \mathbb{N}\})$

↳ If (a_n) is decreasing, then either (a_n) diverges to $-\infty$ or $\lim_{n\to\infty}(a_n) = \inf(\{a_n : n \in \mathbb{N}\})$

The idea of the proof of this theorem (detailed on the next page) is that the restrictions imposed on a sequence by the condition of monotonicity give us enough information about the sequence to conclude that it will diverge to an infinity unless! It levels out, making it converge to a finite value. The proof shows that all cases where the sequence levels out correspond to the sequence being bounded.

The Monotone Convergence Theorem

Proof: Assume that (a_n) is monotonically increasing, & let's suppose first that a_n is <u>not</u> bounded. Then for any $M>0$ there exists some N such that $a_n>M$. But, since (a_n) is montonically increasing, for $n>N$ we have $a_n \geq a_n > M$. Thus, (a_n) diverges to ∞ by definition.

Next, suppose that (a_n) is bounded. Then we have that $\{a_n : n \in \mathbb{N}\}$ is a subset of $\mathbb{R}$ that is bounded above, which by the completeness of $\mathbb{R}$ implies that $\sup(\{a_n : n \in \mathbb{N}\})$ exists. Call this supremum α. We want to show that $\lim_{n\to\infty}(a_n) = \alpha$; that is, we want to show that for any $\varepsilon>0$ there exists some N such that $n>N$ implies $|a_n - \alpha| < \varepsilon$.

To show this, first let $\varepsilon>0$. Since $\sup(\{a_n : n \in \mathbb{N}\}) = \alpha$, by the analytic definition of the supremum there exists some $a_N > \alpha - \varepsilon$. & Since (a_n) is monotonically increasing, we see that for any $n>N$ we have that $a_n \geq a_N > \alpha - \varepsilon$. & of course, $a_n \leq \alpha$ due to the fact that α is the supremum (& hence an upper bound) of $\{a_n : n \in \mathbb{N}\}$. & so for any $n>N$ we have $\alpha - \varepsilon < a_n < \alpha$. This implies that $\alpha - \varepsilon < a_n < \alpha + \varepsilon$, & hence $-\varepsilon < a_n - \alpha < \varepsilon$, which, at last gives $|a_n - \alpha| < \varepsilon$.

The case where (a_n) is monotonically decreasing produces a corresponding result QED

The monotone convergence theorem is useful for several reasons; the first of which is due to the fact that it allows us to show that a sequence converges without knowing its limit. Consider the following example:

Let (a_n) be the sequence where $a_1 = 0.1$, $a_2 = 0.12$, $a_3 = 0.123$, $a_4 = 0.1234$ & so on. Prove that (a_n) converges.

Solution: Note that a_{n+1} & a_n match exactly until the last digits of a_{n+1}, which are the digits of $n+1$. In particular: $a_{n+1} - a_n > 0$ for all n. We have shown that $a_{n+1} > a_n$ for all n, proving that (a_n) is monotone increasing. Furthermore, note that since all the terms begin with 0.1, we have that $a_n \leq 1$ for all n. This makes (a_n) monotone increasing & bounded above, therefore the monotone convergance theorem states that (a_n) must converge! **QED**

Bounded S contains $a_n \Rightarrow \sup(S)$

Theorem: Suppose $S \subseteq \mathbb{R}$ & that S is bounded above. Then there exists a sequence (a_n) where $a_n \in S$ for each n, & $\lim_{n\to\infty}(a_n) = \sup(S)$
Likewise, if S is bounded below then there exists a sequence (b_n), where $b_n \in S$ for each n & $\lim_{n\to\infty}(b_n) = \inf(S)$

Proof: Suppose $S \subseteq \mathbb{R}$ is bounded above, which implies that $\sup(S)$ exists. Let $\sup(S) = \alpha$. By the definition of supremum, for any $\varepsilon > 0$ there exists some $x \in S$ such that $x > \alpha - \varepsilon$. In particular, there exists some $a_1 \in S$ such that $a_1 > \alpha - 1$; & there exists some $a_2 \in S$ such that $a_2 > \alpha - 1/2$; & finally, there exists some $a_3 \in S$ such that $a_3 > \alpha - 1/3$. In general for each $n \in \mathbb{N}$ such that $a_n \in S$ we know there exists such an $a_n > \alpha - 1/n$

Thus, we have a sequence (a_n) from S. Moreover, this sequence does, in fact, converge to α: To show this merely recall that $1/n \to 0$ as $n \to \infty$ & in the trivial case that $(a_n) = (\alpha, \alpha, \alpha, \cdots, \alpha)$ it still converges to α.

Therefore, by the second limit law we have $\alpha - 1/n \to \alpha - 0 = \alpha$ so we are left with $\alpha - 1/n \leq a_n \leq \alpha$ where both the lower & upper bound both converge to α. By the sequence squeeze theorem we see that $\lim_{n\to\infty}(a_n) = \alpha$ QED

Proof of the lower bounded case proceds similarly

Subsequences

Consider the sequence $(a_n) = (n^2)$ or $(1, 4, 9, 16, \ldots)$

$(a_{2n}) = (4, 16, 36, 64, 100, \ldots)$

$(a_{4n-3}) = (1, 25, 81, 169, 289, \ldots)$

Let (a_n) be a sequence of real numbers & let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of intergers. Then $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is called a **subsequence** of a_n & is denoted (a_{n_k})

↳ Note: each term in a subsequence must be unique

Theorem: A sequence (a_n) converges to a if & only if every subsequence of (a_n) also converges to a

Proof: Assume that (a_n) converges to a & that (a_{n_k}) is a subsequence of (a_n). Let $\varepsilon > 0$, then there exists some N such that $(\star)$ $|a_n - a| < \varepsilon$ for all $n > N$. We want to show that $a_{n_k} \to a$. That is, we want to show that there exists some N_1 such that $|a_{n_k} - a| < \varepsilon$ for all $k > N_1$. Notice that since each $n_i \in \mathbb{N}$ & $n_1 < n_2 < n_3 < \cdots$ we have that $n_k \geq k$. Therefore, by letting $N_1 = N$, for any $k > N_1$ we know that $n_k > N$, so by $(\star)$ we have $|a_{n_k} - a| < \varepsilon$.

If seperately we assume that $|a_{n_k} - a| < \varepsilon$ for all $\varepsilon > 0$. Then, we know that $|a_{n_k} - a| < \varepsilon$ for some N_1 such that $n > N_1$. If we then allow N such that $N_1 < N$, then there are some values of N_1 & N that make $|a_n - a| < \varepsilon$ **QED**

Corollary: Different subsequential limits $\Rightarrow$ Diverges.

Functions with Special Names

Let $f: A \to B$ be a function from A to B
The function f is said to be **Injective** if for any $x \neq y$, it produces $f(x) \neq f(y)$.
The function f is said to be **Surjective** if the range $R(f) = B$
Finally, if f is both an injection & a surjection then it is said to be a **bijection**

To prove that f is injective we have to show that for all $x, y \in A$ if $f(x) = f(y)$ then $x = y$

To prove that f is surjective we have to show that for all $b \in B$ there exists some $x \in A$ such that $f(x) = b$

Note: in linear algebra we called injective functions "one-to-one" & surjective functions "onto"

Axiom: (Well-ordering of $\mathbb{N}$) Every non-empty subset of $\mathbb{N}$ has a least element. That is,
$\forall S \subseteq \mathbb{N}, (S \neq \emptyset \to \exists m \in S \ \forall n \in S (m \leq n))$

Theorem: Mathematical Induction

Let $S \subseteq \mathbb{N}$ satisfy the following:

1. $1 \in S$
2. $\forall k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$

Then $S = \mathbb{N}$

Proof: The proof is by contradiction. Suppose that $S \neq \mathbb{N}$. Then $\mathbb{N} \setminus S$ is non-empty $\Rightarrow \exists m \in \mathbb{N} \setminus S$ such that $m \leq k$, $\forall k \in \mathbb{N} \setminus S$. By 1., $m \neq 1$, so $m > 1$ & $m-1 \in \mathbb{N}$. Since $m-1 < m$ & m is the least element of $\mathbb{N} \setminus S \Rightarrow m-1 \in S$. By 2., $k := m-1 \in S$ $\Rightarrow k+1 = (m-1)+1 = m \in S \leftarrow$ This is a contradiction! QED

Thus, $S = \mathbb{N}$

The above theorem is rigorous, but often too abstract, the following is a more useful version of the induction principle

Let $P(n)$ be a statement & suppose that

1. $P(1)$ is true
2. $\forall k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Mathematical Induction Example: Prove that

$1+2+\cdots+n=\frac{1}{2}n(n+1)$, $\forall n \in \mathbb{N}$.

Proof: Let $P(n)$: "$1+2+\cdots+n=\frac{1}{2}n(n+1)$"

<u>Step 1:</u> $P(1)$: "$1=\frac{1}{2}1(2)$" is true ✓

<u>Step 2:</u> Fix $k \in \mathbb{N}$ & assume $P(k)$ is true.

This is called the induction hypothesis

We assume that "$1+2+\cdots+k=\frac{1}{2}k(k+1)$" is true.

<u>Step 3:</u> Show that $P(k+1)$ is true.

$$(1+2+\cdots+k)+(k+1)=\frac{1}{2}k(k+1)+(k+1) \quad \text{by step 2}$$

$$=\frac{1}{2}(k+1)(k+2) \quad ✓$$

Thus, the formula is true $\forall n \in \mathbb{N}$ by mathematical induction. QED

It maybe the case that $P(n)$ is false until n is sufficiently large but is true for all $n > n_0$ for some n_0. Mathematical induction can be modified to prove such statements

Modified Mathematical Induction

Let $n_0 \in \mathbb{N}$ & let $P(n)$ be a statement for each $n \in \mathbb{N} : n \geq n_0$. Suppose that

1. The statement $P(n_0)$ is true
2. For all $k \geq n_0$, the truth of $P(k)$ implies that $P(k+1)$ is also true

Then, $P(n)$ is true for all $n \geq n_0$

Another Mathematical Induction Example

Prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 $\forall n \in \mathbb{N}$ using mathematical induction.

Proof: Let $f(n)=n^3+(n+1)^3+(n+2)^3$. This can be expanded where $(n+1)^3=n^3+3n^2+3n+1$ & $(n+2)^3=n^3+6n^2+12n+8$

$\therefore f(n)=n^3+(n^3+3n^2+3n+1)+(n^3+6n^2+12n+8)$

$=3n^3+9n^2+15n+9=3(n^3+3n^2+5n+3)$

$\therefore f(1)=1^3+2^3+3^3=1+8+27=36=4\times 9$ ✓ (base case)

Assume that for some $k \in \mathbb{N}$. $\frac{3k^3+9k^2+15k+9}{9} \in \mathbb{N}$

We can then write:

$f(k+1)=f(k)+[f(k+1)-f(k)]$ ⟵ If we can show that this difference is divisible by 9 then since $f(k)$ is divisible by 9 we know that $f(k+1)$ must also be divisible by 9. (induction hypothesis)

$f(k+1)-f(k)=[\cancel{(k+1)^3}+\cancel{(k+2)^3}+(k+3)^3]-[k^3+\cancel{(k+1)^3}+\cancel{(k+2)^3}]$

$=(k+3)^3-k^3$

$=(\cancel{k^3}+9k^2+27k+27)-\cancel{k^3}$

$=9k^2+27k+27=9(k^2+3k+3)$ ✓

Thus, $f(k+1)-f(k)$ is divisible by 9 meaning that $f(k+1)$ is also divisible by 9. This means that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$ by mathematical induction. **QED**

Countability & Infinite Sets

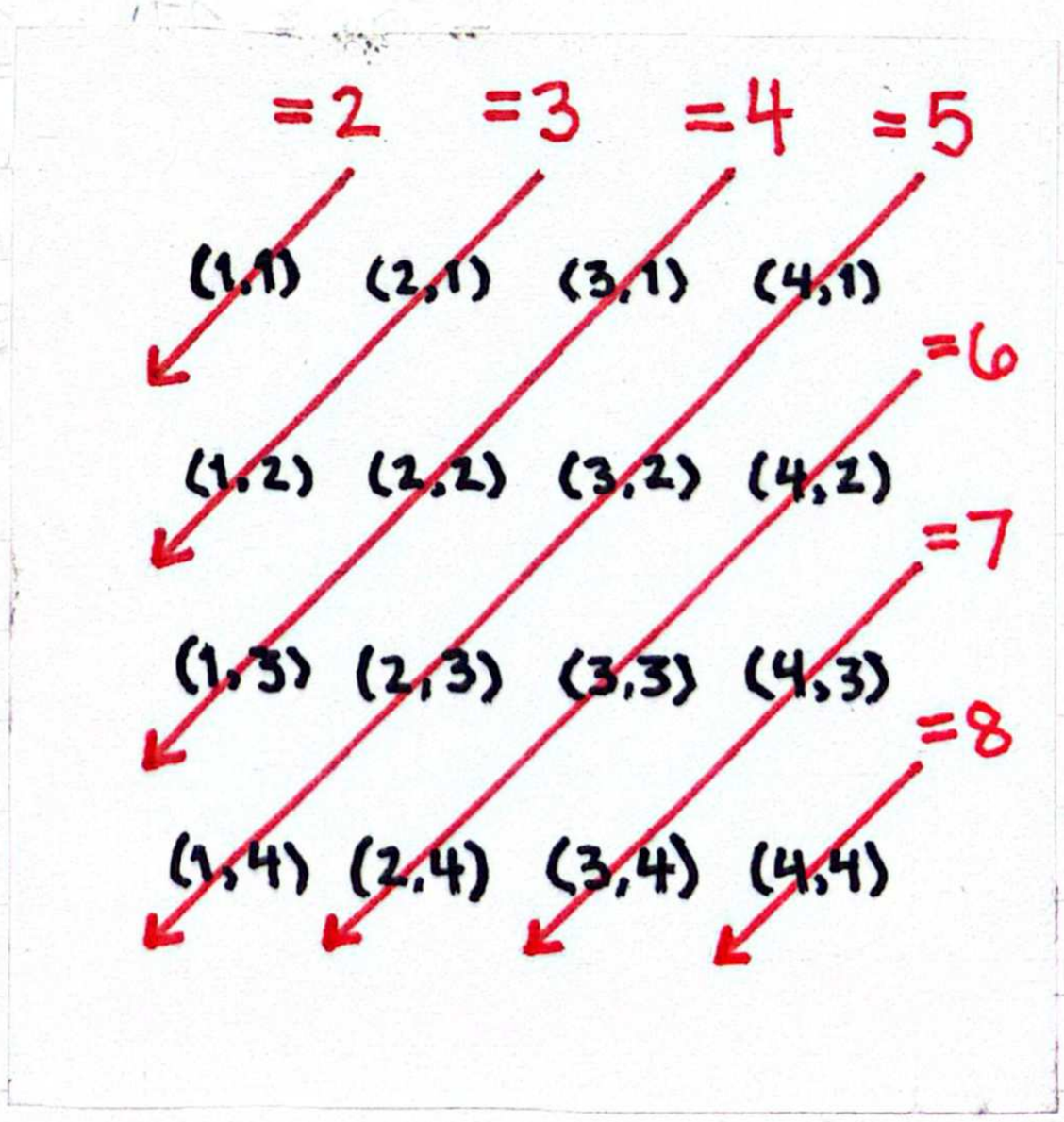
Not all infinities are the same size. The nicest infinities that behave the most intuitively are called *Countable Infinities*.

A set S is said to be **denumerable** (or *Countably Infinite*) if there exists a bijection of $\mathbb{N}$ onto S.

↳ A set is said to be **countable** if it is either finite <u>or</u> denumerable.

Theorem: The set $\mathbb{N} \times \mathbb{N}$ is denumerable.

The set $\mathbb{N} \times \mathbb{N}$ is the set of all ordered pairs (m, n) where $m, n \in \mathbb{N}$. The idea of Cantor's *Winding Bijection* is to create a grid of these ordered pairs & then by traversing this grid along the anti-diagonals where each entry in a given anti-diagonal has the same value when their $m+n$ are summed. This creates a denumerable number of sets, each with a denumerable number of elements. The fact that $\mathbb{N} \times \mathbb{N}$ can be expressed this way means it must be denumerable. **QED**



Countable Union of Countable Sets is countable

Theorem: If A_m is a countable set for each $m \in \mathbb{N}$, then, the union $A := \bigcup_{=1}^{\infty} A_m$ is countable.

Proof: for each $m \in \mathbb{N}$ let φ_m be a surjection of $\mathbb{N}$ onto A_m. We can then define $\beta : \mathbb{N} \times \mathbb{N} \to A$ as $\beta(m,n) := \varphi_m(n)$

Since if $a \in A$ there exists a least $m \in \mathbb{N}$ such that $a \in A_m$ (meaning that there exists a least $n \in \mathbb{N}$ such that $a = \varphi_m(n)$) we know that β is surjective.

Thus, by the theorem on the previous page, since we can map β onto $\mathbb{N} \times \mathbb{N}$ & then map that onto $\mathbb{N}$, meaning that A must be countable.

QED

To begin our scrutinus of the real numbers it will be useful to define some of their keu properties

Algebraic Properties of $\mathbb{R}$: *(The Field Axioms)*

A1. $a+b=b+a$

A2. $(a+b)+c=a+(b+c)$

A3. $\exists 0 \in \mathbb{R}$ such that $\forall a \in \mathbb{R}$ $0+a=a$

A4. $\forall a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a+(-a)=0$

M1. $a \cdot b = b \cdot a$

M2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M3. $\exists 1 \in \mathbb{R}$ such that $a \cdot 1 = a$

M4. For all $a \neq 0 \in \mathbb{R}$ $\exists 1/a \in \mathbb{R}$ such that $a \cdot 1/a = 1$

D. $a \cdot (b+c) = a \cdot b + a \cdot c$

Anu set $\mathbb{F}$ with two binaru operations $+, \cdot$ that satisfu these properties, denoted $(\mathbb{F}, +, \cdot)$ is said to be a **Field.** The realnumbers $\mathbb{R}$ with addition $+$ & multiplication $\cdot$ are an example of a field: $(\mathbb{R}, +, \cdot)$

Theorem: (Uniqueness of 0 & 1)

1. If $a, b \in \mathbb{F}$ & $a+b=b \Rightarrow a=0$
2. If $a, b \in \mathbb{F}$ & $a \cdot b = b \Rightarrow a=1$
3. If $a \in \mathbb{F} \Rightarrow a \cdot 0 = 0$

Proof:

1.:

$$\begin{aligned} a &= a+0 && \text{by A3} \\ &= a+(b+(-b)) && \text{by A4} \\ &= (a+b)+(-b) && \text{by A2} \\ &= b+(-b) && \text{by assumption} \\ &= 0 && \text{by A4} \end{aligned}$$

2.:

$$\begin{aligned} a &= a \cdot 1 && \text{by M3} \\ &= a \cdot (b \cdot (1/b)) && \text{by M4} \\ &= (a \cdot b) \cdot (1/b) && \text{by M2} \\ &= b \cdot (1/b) && \text{by assumption} \\ &= 1 && \text{by M4} \end{aligned}$$

3.:

$$\begin{aligned} a + a \cdot b &= a \cdot 1 + a \cdot 0 && \text{by M3} \\ &= a \cdot (1+0) && \text{by D} \\ &= a \cdot 1 && \text{by A3} \\ &= a && \text{by M3} \\ a \cdot 0 &= 0 && \text{by 1.} \end{aligned}$$

QED

Theorem: (Uniqueness of Inverse)

1. If $a \neq 0$ & $a \cdot b = 1$, then $b = 1/a$
2. If $a \cdot b = 0$ then either $a = 0$ or $b = 0$

Proof:

1.
$$
\begin{aligned}
b &= 1 \cdot b && \text{by M1, M3} \\
&= (1/a \cdot a) \cdot b && \text{by M1, M4} \\
&= 1/a \cdot (a \cdot b) && \text{by M2} \\
&= 1/a \cdot 1 && \text{by assumption} \\
&= 1/a && \text{by M3}
\end{aligned}
$$

2. If $a = 0$ the result is immediate

If $a \neq 0$, then $\exists\, 1/a$ — by M4

$$
\begin{aligned}
\Rightarrow b &= 1 \cdot b && \text{by M1, M3} \\
&= (1/a \cdot a) \cdot b && \text{by M1, M4} \\
&= 1/a \cdot (a \cdot b) && \text{by M2} \\
&= 1/a \cdot 0 && \text{by assumption} \\
&= 0 && \text{by 3 of the Previous theorem}
\end{aligned}
$$

QED

Order Properties of $\mathbb{R}$: Let $\mathbb{F}$ be a field. A subset P of $\mathbb{F}$ are called positive numbers if they satisfy the following:

P1. if $a, b \in P$, then $a+b \in P$

P2. if $a, b \in P$, then $a \cdot b \in P$

P3. if $a \in \mathbb{F}$ then exactly one of the following holds: $a \in P$ or $a = 0$ or $-a \in P$

This is referred to as the *Trichotomy Property* of ordered fields.

Any field that satisfies these three conditions is said to be an **Ordered Field**

$$a \in \mathbb{F} \text{ is } \begin{bmatrix} \text{positive} \\ \text{negative} \\ \text{nonpositive} \\ \text{non negative} \end{bmatrix} \text{ if } \begin{bmatrix} a \in P \\ -a \in P \\ a \in P \cup \{0\} \\ -a \in P \cup \{0\} \end{bmatrix}$$

we say $a < b$ or $b > a$ if $b - a \in P$... alternatively, we say $a \leq b$ or $b \geq a$ if $b - a \in P \cup \{0\}$

Theorem 2.4: Let $a, b, c \in \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then:

1. If $a > b$ & $b > c$ then $a > c$
2. If $a > b$ then $a + c > b + c$
3. If $a > b$ & $c > 0$ then $c \cdot a > c \cdot b$, If $c < b$ then $c \cdot a < c \cdot b$

Proof

1. $a - b,\ b - c \in P \Rightarrow (a-b) + (b-c) = a - c \in P$
2. $(a+c) - (b+c) = a - b \in P$
3. If $c \in P \Rightarrow c \cdot (a-b) = c \cdot a - c \cdot b \in P$
   If $-c \in P \Rightarrow -c \cdot (a-b) = c \cdot b - c \cdot a \in P$

QED

Absolute Value & the Real Line

Let $\mathbb{F}$ be an ordered field, then we can define the **Absolute Value** of $a \in \mathbb{F}$, denoted by $|a|$, as follows

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem 2.7: (Properties of absolute value)

1. $|ab| = |a| \cdot |b|$
2. $|a|^2 = a^2$
3. If $c \geq 0$, then $|a| \leq c$ iff $-c \leq a \leq c$
4. $-|a| \leq a \leq |a|$

Bernoulli's Inequality: $x > -1 \Rightarrow (1+x)^n \geq 1 + n + x$

Arithmatic Mean – Geometric Mean Inequality: If $a, b > 0$, then we have the following : $\sqrt{ab} \leq \frac{1}{2}(a+b)$

Proof: If $a = b$, then

$\sqrt{ab} = a = \frac{1}{2}(a+b) \Rightarrow 4ab = (a+b)^2 = a^2 + 2ab + b^2$

$\Longleftrightarrow (a-b)^2 = 0 \Rightarrow a = b$

If $a \neq b \Rightarrow \sqrt{a} \neq \sqrt{b}$ & $(\sqrt{a} - \sqrt{b})^2 > 0 \Rightarrow a + b - 2\sqrt{ab} > 0$

Theorem 2.8: The triangle inequality
Let $\mathbb{F}$ be an ordered field. Then $\forall a, b \in \mathbb{F}$ we have $|a+b| \leq |a|+|b|$

Proof:

From $-|a| \leq a \leq |a|$ & $-|b| \leq b \leq |b|$, we get $-(|a|+|b|) \leq a+b \leq |a|+|b|$. Thus, by theorem 2.7.③, $|a+b| \leq |a|+|b|$ QED

Corollary 1: Let $\mathbb{F}$ be an ordered field & $a, b \in \mathbb{F}$

① $||a|-|b|| \leq |a-b|$

② $|a-b| \leq |a|+|b|$

Proof:

① Replace a by $a-b$

$\Rightarrow |a| = |(a-b)+b| \leq |a+b|+|b|$ by thm 2.8

$\Rightarrow |a|-|b| \leq |a-b|$

Similarly, replace b by $b-a$

$\Rightarrow |b|-|a| \leq |b-a| = |a-b|$

② Replace b by $-b$ in thm 2.8 QED

Corollary 2: let $a_1, \ldots, a_n \in \mathbb{F}$ where $\mathbb{F}$ is an ordered field, then, $|a_1+\cdots+a_n| \leq |a_1|+\cdots+|a_n|$

Proof: Just apply mathematical induction to theorem 2.8 QED

Determine the set $A = \{x \in \mathbb{R} : |x-1| < 2|x^2|\}$

Method 1: When does the absolute value equal zero? $\longrightarrow$ when $x=1$ or $x=0$

Case 1: $x \geq 0 \Rightarrow x-1 < 2x \Rightarrow x > -1/2$, so $x \geq 1$

Case 2: $0 \leq x < 1 \Rightarrow -(x-1) < 2x \Rightarrow 1/3 < x < 1$

Case 3: $x < 0 \Rightarrow -(x-1) < -2x \Rightarrow$ so $x < -1$

Method 2: Using $a<b$ iff $a^2 < b^2$ when $a,b \geq 0$

$\Rightarrow |x+1| < 2|x|$ iff $|x+1|^2 < (2|x|)^2$. Since $|a|^2 = a^2$ we get $(x-1)^2 < 4x^2$

$\Rightarrow 3x^2 + 2x - 1 > 0 \rightarrow (3x-1)(x+1) > 0$

$\Rightarrow 3x-1 > 0$ & $x+1 > 0 \rightarrow x > 1/3$ & $x < -1$

We can use $|b-a|$ as the distance between two $a,b \in \mathbb{F}$ (since distance is always positive & the absolute value is too this should feel intuitive)

Using this distance we can define the ε-neighborhood around any $a \in \mathbb{F}$ as follows:

$$V_\varepsilon = \{x \in \mathbb{F} : |x-a| < \varepsilon\}$$

Theorem 2.9:

Let $a \in \mathbb{F}$ where $\mathbb{F}$ is an ordered field. If $x \in V_\varepsilon(a)$ for all $\varepsilon > 0$, then $x = a$.

↳ This follows as a result of theorem 2.7

Supremums & Their Properties:

Assume that $S \subseteq \mathbb{F}$, $S \neq \emptyset$ & that $\mathbb{F}$ is an ordered field

Lemma 1: $u = \sup S \Leftrightarrow u$ satisfies:

a. $\forall s \in S \quad s \leq u$ *(all s in S less than or equal to u)*

b. $v < u \Rightarrow \exists s' \in S : v < s'$ *(if v is less than u then there exists a s' in S such that s' is less than v)*

Lemma 2: An upper bound u of S is the supremeum of S if & only if for all $\varepsilon > 0$ there exists a s in S such that $u - \varepsilon < s$

Example: Find the supremum of $H = \{x \in \mathbb{R} : x < \frac{1}{x}\}$

If $x = 0 \rightarrow$ undefined

if $x > 0$, multiplys by x: $x^2 > 1 \rightarrow 0 < x < 1$

if $x < 0$, multiply by x: $x^2 > 1 \rightarrow x < -1$

Therefore, $H = (-\infty, -1) \cup (0, 1)$

1 is an upper bound of H

For all $x \in (-\infty, -1)$ $x < 1$ since 1 is positive

furthermore, $x < 1$ for all $x \in (0, 1)$.

No smaller number is an upper bound

Fix $u < 1$. Define $\varepsilon := 1 - u > 0$.

Set $x := 1 - \varepsilon/2 = (u+1)/2$. Then:

$0 < x < 1$ (since $u < x < 1$) $\Rightarrow x \in (0, 1) \subseteq H$

$x > u$ (since $x - u = (1-u)/2 = \varepsilon/2 > 0$)

$\Rightarrow u$ is not an upper bound.

Thus, since 1 is an upper bound & no lesser number is an upper bound we can conclude that 1 is the supremum of the set H.

Assume that $S \subseteq \mathbb{R}$ & $a \in \mathbb{R}$
Define $a+S := \{a+s : s \in S\}$. More generally, let $A, B \subseteq \mathbb{R}$ define $A+B := \{a+b : a \in A,\ b \in B\}$

Theorem 2.9: *Archimedean Property*
$\forall x \in \mathbb{R},\ \exists n_x \in \mathbb{N}$ such that $x < n_x$.

Proof: Suppose for a contradiction that for some $x \in \mathbb{R}$ $\neg \exists n_x \in \mathbb{N} : x < n_x$.
$\Rightarrow \exists x \in \mathbb{R}$ such that $\forall n \in \mathbb{N},\ n \leq x$
$\Rightarrow x$ is an upper bound of $\mathbb{N}$.
By completeness, $\mathbb{N} (\neq \emptyset)$ has a supremum in $\mathbb{R}$. Let $u = \sup \mathbb{N} \leq x$. But since $u-1 < u$, $u-1$ is not an upper bound of $\mathbb{N}$.
$\Rightarrow \exists n \in \mathbb{N}$ such that $u-1 < n$
$\Rightarrow u < n+1 \in \mathbb{N} \rightarrow$ But this is a contradiction since u is not an upper bound it is not a supremum. **QED**

Corollary 1: Let $S := \{1/n : n \in \mathbb{N}\}$. Then $\inf S = 0$
Proof: Suppose that $S \neq \emptyset$ & S is bounded below.
$\Rightarrow \exists \inf S$, let $w := \inf S$ & clearly $w \geq 0$.
$\forall \varepsilon > 0$, by the archimedean property, $\exists n \in \mathbb{N}$ such that $1/\varepsilon < n \Longleftrightarrow 1/n < \varepsilon$. Then $0 \leq w \leq 1/n < \varepsilon$. Since ε is arbitrary, by theorem 2.9 $\inf S = 0$.

Corollary 2: If $t > 0$, $\exists n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.

Corollary 3: If $y>0$, $\exists n_y \in \mathbb{N}$ such that $n_y - 1 \leq y \leq n_y$.

Proof. Let $A_y := \{m \in \mathbb{N} : y < m\} \subseteq \mathbb{N}$. By the Archimedean property, A_y is non-empty. So, by the well-ordering property of $\mathbb{N}$, $\exists a \in A$ such that a is the least element in A. Thus, $a-1 \notin A_y$. Hence, $a-1 \leq y \leq a$. **QED**

Existence of the Square Root of 2

Theorem 2.10: existence of $\sqrt{2}$
There exists a positive real number x such that $x^2 = 2$.

Proof:

We need to show the following:
1. $S := \{x \in \mathbb{R} : x^2 < 2\}$ is non-empty.
2. S is bounded above.
3. $x := \sup(S)$ satisfies $x^2 = 2$

1. $1^2 < 2 \Rightarrow 1 \in S$ ✓
2. 2 is an upper bound of S since $\forall s > 2$,
$s^2 > 2^2 = 4 > 2 \Rightarrow s \notin S$ ✓

Hence, $\exists x := \sup(S) \in \mathbb{R}$ & $x > 1$

3. Assume, for a contradiction, that $x^2 < 2$

$$\frac{1}{n^2} \leq \frac{1}{n} \Rightarrow \left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{1}{n}(2x+1)$$

If $x^2 + \frac{1}{n}(2x+1) < 2$, then our work is done.

$$\Leftrightarrow \frac{1}{n}(2x+1) < 2 - x^2$$

$$\Leftrightarrow \frac{1}{n} < \frac{2-x^2}{2x+1}$$ By assumption, $2 - x^2 > 0$

$$\Rightarrow \frac{2-x^2}{2x+1} > 0$$ By the archimedean property, $\exists n \in \mathbb{N}$ to satisfy this.

Thus, we have shown that $\left(x + \frac{1}{n}\right)^2 < 2$ hence, $x + \frac{1}{n} \in S$ is a contradiction, making $x^2 < 2$ impossible. ✓

The final step is to show that there exists some $m \in \mathbb{N}$ such that $x - 1/m$ is an upper bound of S.

③ (cont...) Assume, for a contradiction, that $x^2 > 2$. Note also that

$$\left(x-\frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

$$x^2-2 > \frac{2x}{m} \iff \frac{1}{m} < \frac{x^2-2}{2x} \iff x^2 - \frac{2x}{m} > 2$$

By assumption, $x^2-2>0$, so $\frac{x^2-2}{2x}>0$. By the Archimedean property $\exists m \in \mathbb{N}$ that satisfies the above, meaning $\left(x-\frac{1}{m}\right)^2>0$. Now, $\forall s \in S$,

$s^2 < 2 < \left(x-\frac{1}{m}\right)^2$ hence, $s < x - \frac{1}{m}$, so

$x-\frac{1}{m}$ is an upper bound of S, but this is a contradiction, so $x^2 > 2$ is not possible.

Finally, by the trichotomy property we have no choice but to conclude that $x^2 = 2$.

QED

Intervals

Theorem 2.11: If $S \subseteq \mathbb{R}$ satisfies the following:
$\forall x, y \in S$ such that $x < y \Rightarrow [x, y] \subseteq S$
Then S is an interval.

Proof: Either S is bounded or it is unbounded.

Case 1: S is bounded

Let $a = \inf(S)$ & $b = \sup(S)$ define $z \in \mathbb{R}$ such that $a < z < b$.

$\Rightarrow z \neq \inf(S)$, $\exists x \in S$ such that $x < z$

$\Rightarrow z \neq \sup(S)$, $\exists y \in S$ such that $y > z$

$\Rightarrow z \in [x, y] \subseteq S$, hence $(a, b) \subseteq S$

Case 2: S is unbounded

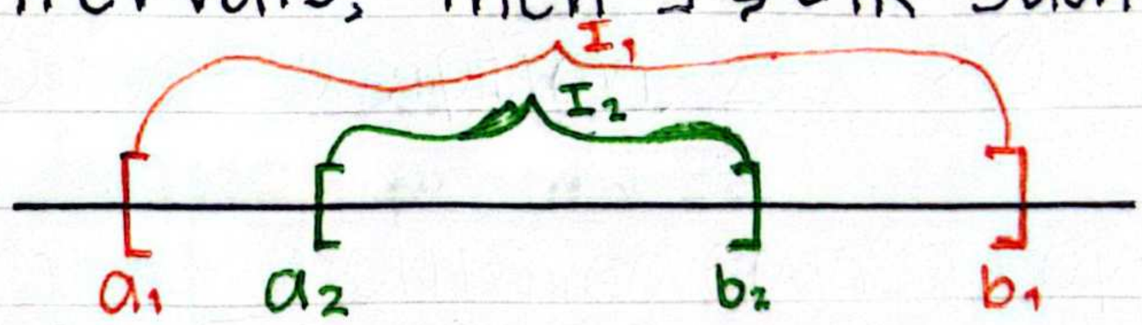
Replace either a or b with $-\infty$ & ∞ above.

In either case, S is an interval satisfying the theorem.

QED

Theorem 2.12: nested interval property

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed, bounded intervals, then $\exists \xi \in \mathbb{R}$ such that $\xi \in I_n$ $\forall n \in \mathbb{N}$



Proof: $\{I_n\}$ is nested

$\Rightarrow I_n \subseteq I_1$ so, $a_n \leq b_1$ $\forall n \in \mathbb{N}$

$\Rightarrow$ The set $\{a_n : n \in \mathbb{N}\}$ is bounded above.

$\Rightarrow$ Let $\xi := \sup(\{a_n\}) \Rightarrow$ Clearly, $a_n \leq \xi$ $\forall n \in \mathbb{N}$

Next, since $I_n \subseteq I_k$ for any k we get

$a_k \leq a_n \leq b_n$ & $a_n \leq b_k \leq b_n \Rightarrow \xi \leq b_n$.

Therefore, $a_n \leq \xi \leq b_n$ for every n, which implies that $\xi \in I_n$ $\forall n \in \mathbb{N}$.

QED

Theorem 2.13: If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals such that lengnths $b_n - a_n$ of I_n satisfy $\inf(\{b_n - a_n : n \in \mathbb{N}\}) = 0$ Then their exists a unique $\xi \in \mathbb{R}$ such that $\xi \in I_n$ $\forall n \in \mathbb{N}$

Proof: $\exists \eta$ such that $\eta \in I_n$ $\forall n \in \mathbb{N}$, since $a_n \leq \eta$ $\forall n \in \mathbb{N}$ & $\xi = \sup(\{a_n\}) \Rightarrow \xi \leq \eta$.
Since $a_n \leq \xi \leq \eta \leq b_n$ $\forall n \in \mathbb{N} \Rightarrow 0 \leq \eta - \xi \leq b_n - a_n$.
From $\inf(\{b_n - a_n\}) = 0 \Rightarrow \exists m \in \mathbb{N} : b_m - a_m < \varepsilon$ $\forall \varepsilon > 0$
Then $0 \leq \eta - \xi < \varepsilon$ $\forall \varepsilon > 0$. By Theorem $\eta - \varepsilon = 0$.
that is, ξ is the onlu point in $\bigcap_{n=1}^{\infty} I_n$ **QED**

We can rigorously prove that $\mathbb{N}\times\mathbb{N}$ is countable

$$\Psi(k) = 1+2+\cdots+k = \tfrac{1}{2}k(k+1)$$

$$h(m,n) = \Psi(m+n-2)+m$$

<u>Goal</u>: Show that $h:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is a bijection

h is injective

$(m,n)\neq(m',n') \Rightarrow$ either $\begin{cases} \text{i. } m+n\neq m'+n' \\ \text{ii. } m+n=m'+n' \ \& \\ \quad m\neq m' \end{cases}$

i. suppose that

$m+n<m'+n'$, then, since Ψ is increasing & $m'>0$

$$\begin{aligned} h(m,n) &= \Psi(m+n-2)+m \leq \Psi(m+n-2)+(m+n-1) \\ &= \Psi(m+n-1) \leq \Psi(m'+n'-2) \\ &< \Psi(m'+n'-2)+m' = h(m',n') \end{aligned}$$

ii. $m+n=m'$ & $m\neq m'$ implies that

$$h(m,n)-m = \Psi(m+n-2) = \Psi(m'+n'-2) = h(m',n')-m'$$

Thus, in either case, $h(m,n)\neq h(m',n')$

h is surjective

Clearly $h(1,1)$. If $p\in\mathbb{N}$ with $p\geq 2$, we will find a pair $(m_p,n_p)\in\mathbb{N}\times\mathbb{N}$ where $h(m_p,n_p)=p$. Since $p<\Psi(p)$, the set $E_p := \{k\in\mathbb{N} : p\leq\Psi(k)\}$ is non-empty. Using the well-ordering property, we let $k_p>1$ be the least element in E_p

(this means that p lies in the k_p^{th} diagonal)

Since $p\geq 2$, it follows that:

$$\Psi(k_p-1) < p < \Psi(k_p) = \Psi(k_p-1)+k_p$$

Let $m_p := p-\Psi(k_p-1)$ so that $1\leq m_p\leq k_p$ &

$n_p := k_p-m_p+1$ so that $1\leq n_p\leq k_p$ &

$m_p+n_p-1 = k_n$. Therefore,

$$\begin{aligned} h(m_p,n_p) &= \Psi(m_p+n_p-2)+m_p \\ &= \Psi(k_p-1)+m_p = p \end{aligned}$$

Theorem 3.1: A convergent sequence in $\mathbb{R}$ is bounded, meaning $\exists M>0 : |x_n| \leq M \quad \forall n \in \mathbb{N}$

Proof: let $x_n \to x$ & fix $\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon \quad \forall n \geq N$.

$$\Rightarrow |x_n| = |x_n + x - x| \leq |x_n - x| + |x| < |x| + \varepsilon \quad \forall n > N$$

Set $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x| + \varepsilon\}$

$$\Rightarrow |x_n| \leq M \quad \forall n \in \mathbb{N}$$

How do we prove that $(x_n) \to x$?

1. "Let $\varepsilon > 0$ be arbitrary"
2. Demonstrate a choice of $N \in \mathbb{N}$.
   (This part usually requires the most work)
3. Now show that N actually works.
4. "Assume $n \geq N$"
5. With N well chosen derive $|x_n - x| < \varepsilon$

Monotone Convergence Theorem:

A monotone sequence of real numbers is convergent if & only if it is bounded. Furthermore:

1. If (x_n) is *increasing* & bounded above, then $\lim(x_n) = \sup(x_n : n \in \mathbb{N})$
2. If (x_n) is *decreasing* & bounded below, then $\lim(x_n) = \inf(x_n : n \in \mathbb{N})$

Proof:

($\Rightarrow$) Let $(x_n) \to x$. Fix $\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N} : |x_n - x| < \varepsilon \ \forall n \geq N$

$\Rightarrow |x_n| = |x_n - x + x| \leq |x_n - x| + |x| < |x| + \varepsilon \quad \forall n \geq N$

Set $M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |x| + \varepsilon\}$

$\Rightarrow |x_n| \leq M \ \forall n \in \mathbb{N}$

($\Leftarrow$) 1. Let (x_n) be increasing & bounded above $\Rightarrow \exists x^* = \sup\{x_n : n \in \mathbb{N}\}$. To show that $x^* = \lim(x_n)$

$\boxed{\text{take } \varepsilon > 0} \Rightarrow x^* - \varepsilon$ is <u>not</u> an upper bound of $\{x_n : n \in \mathbb{N}\}$, $\boxed{\exists N \in \mathbb{N} :} x_N > x^* - \varepsilon$. Since (x_n) is increasing $\forall n \geq N \quad x^* - \varepsilon < x_N \leq x_n \leq x^* < x^* + \varepsilon$

$\therefore \boxed{|x_n - x^*| < \varepsilon \ \forall n \geq N}$

Thus, since ε is chosen arbitrarily (by $\square$) $x_n \to x^*$

2. Let (x_n) be decreasing & bounded below $\Rightarrow \exists x^* = \inf\{x_n : n \in \mathbb{N}\}$ To show that $x^* = \lim(x_n)$

$\boxed{\text{take } \varepsilon > 0} \Rightarrow x^* + \varepsilon$ is <u>not</u> a lower bound of $\{x_n : n \in \mathbb{N}\}$, $\boxed{\exists N \in \mathbb{N} :} x_N < x^* + \varepsilon$. Since (x_n) is decreasing $\forall n \geq N \quad x^* + \varepsilon > x_N \geq x_n \geq x^* > x^* - \varepsilon$

$\therefore \boxed{|x_n - x^*| < \varepsilon \ \forall n \geq N}$

Thus, since ε is chosen arbitrarily (by $\square$) $x_n \to x^*$

QED

Example: Let (x_n) be defined bus $x_1 = \frac{1}{2}$, $x_{n+1} = \frac{3}{2} x_n(1-x_n)$
Is (x_n) convergent? If so find $\lim(x_n)$

Assume $x_n \to x \Rightarrow \lim(x_{n+1}) = x = \lim(x_n)$

$\Rightarrow x = \frac{3}{2} x(1-x) \Rightarrow 3x^2 - x = 0$ <u>or</u> $x(3x-1) = 0$

So, either $x = 0$ or $x = 1/3$. Thus, we want to show that $1/3 < x_{n+1} < x_n \leq 1/2$. We can use mathematical induction: $x_2 = \frac{3}{2}(\frac{1}{2})(1-\frac{1}{2}) = \frac{3}{8}$

$n=1$: $\frac{1}{3} < x_2 = \frac{3}{8} < \frac{1}{2} = x_1 \leq \frac{1}{2}$

Assume that for some k $\frac{1}{3} < x_{k+1} < x_k \leq \frac{1}{2}$

for $x_{k+2} = \frac{3}{2} x_{k+1}(1-x_{k+1})$

$$< \frac{3}{2} x_{k+1} \cdot \frac{2}{3} = x_{k+1}$$

To show $x_{k+2} > 1/3$,

$$x_{k+2} - \frac{1}{3} = \frac{-3}{2} x_{k+1}^2 + \frac{3}{2} x_{k+1} - \frac{1}{3}$$

$$= \frac{-3}{2}\left(x_{k+1}^2 - x_{k+1} + \frac{2}{9}\right)$$

$$= \frac{-3}{2}\left(\left(x_{k+1} - \frac{1}{2}\right)^2 - \frac{1}{36}\right) > 0$$

since $\frac{1}{3} < x_{k+1} < \frac{1}{2} \Rightarrow |x_{k+1} - \frac{1}{2}| < \frac{1}{6}$

Thus, $\frac{1}{3} < x_{n+1} < x_n \leq \frac{1}{2}$ $\forall n \in \mathbb{N}$

Since (x_n) is decreasing, meaning that by the MCT (x_n) is convergent & $\lim(x_n) = 1/3$

Example: Let (z_n) be a sequence in $\mathbb{R}$ given by:

$$z_1 = 1, \quad z_{n+1} = \sqrt{2z_n} \quad \forall n \in \mathbb{N}$$

Show that $\lim(z_n) = 2$.

$z_1 = 1, \; z_2 = \sqrt{2} \Rightarrow 1 \leq z_1 \leq z_2 < 2$

We want to show that $1 \leq z_n < z_{n+1} < 2 \quad \forall n \in \mathbb{N}$ which can be shown via mathematical induction:

Base Case: $n = 1 \Rightarrow z_1 = 1$ ✓

Induction: Assume $1 \leq z_k < z_{k+1} < 2$

$n = k+1$: $1 \leq z_{k+1}$ follows immediately.

$$z_{k+2} = \sqrt{2z_{k+1}} > \sqrt{z_{k+1} \cdot z_{k+1}} = z_{k+1}$$

$$z_{k+2} = \sqrt{2z_{k+1}} < \sqrt{4} = 2$$ ✓

Thus, z_n is bounded & decreasing, meaning by the MCT z_n is convergent

Let $z = \lim(z_n) \Rightarrow z = \lim(z_{n+1}) = \lim(\sqrt{2z_n}) = \sqrt{2z}$

$\Rightarrow z^2 = 2z$ meaning $z = 2$ since $1 \leq z \leq 2$

Therefore, $\lim(z_n) = 2$

QED

Constructing Euler's Number:

Define $e_n = (1+\frac{1}{n})^n$ We will show that $e_n \to e$

Step 1: (e_n) is increasing:

$$e_n = (1+\tfrac{1}{n})^n = 1 + n\frac{1}{n} + \frac{1}{2}n(n-1)\frac{1}{n^2} + \frac{1}{3!}n(n-1)(n-2)\frac{1}{n^3}$$

$$+ \cdots + \frac{n(n-1)\cdots 2\cdot 1}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2}(1-\frac{1}{n}) + \frac{1}{6}(1-\frac{1}{n})(1-\frac{2}{n})$$

$$+ \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

$$e_{n+1} = 1 + 1 + \frac{1}{2}(1-\frac{1}{n+1}) + \cdots + \frac{1}{n!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})$$

$$+ \frac{1}{(n+1)!}(1-\frac{1}{n+1})\cdots(1-\frac{n}{n+1})$$

Since $0 < 1-\frac{k}{n} < 1-\frac{k}{n+1}$ $\forall k = 1,2,3,\ldots,n-1$

we get $e_n < e_{n+1}$.

Step 2: (e_n) is bounded:

using the following: $\forall k = 1,2,\ldots,n-1$ $\begin{cases} 1-\frac{k}{n} < 1 \\ 2^{k-1} \le k! \end{cases}$

$$e_n = 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})\cdots(1-\frac{n-1}{n})$$

$$< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \boxed{\frac{1}{2} + \cdots + \frac{1}{2^{n-1}}} < 3$$

$$= 1 - \frac{1}{2^{n-1}} < 1$$

So, $2 < e_n < 3$

Hence, by the MCT $\lim(e_n) = e$ exists.

Limit Theorems Cont...

Theorem: Let (x_n) be a sequence in $\mathbb{R}$ such that $\lim(x_n) = x$ & $x_n \geq 0$ $\forall n \in \mathbb{N}$. Then $\lim(\sqrt{x_n}) = \sqrt{x}$

Proof: since $\forall n \in \mathbb{N}$ $x_n \geq 0$ we know that $x \geq 0$.

(i) $x = 0 \Rightarrow$ let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$0 \leq x_n = x_n - 0 < \varepsilon^2 \Rightarrow 0 \leq \sqrt{x_n} < \varepsilon$

(ii) $x > 0 \Rightarrow \sqrt{x} > 0$

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$, it follows that

$$|\sqrt{x_n} - \sqrt{x}| \leq \left(\frac{1}{\sqrt{x}}\right)|x_n - x|$$

Thus $x_n \to x \Rightarrow \sqrt{x_n} \to \sqrt{x}$ **QED**

The **m-tail** of a sequence (x_n) is the sequence defined bu $x_m = (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \ldots)$

Theorem: Let (x_n) be a sequence in $\mathbb{R}$ & $m \in \mathbb{N}$ The m-tail of (x_n) converges if & onlys if (x_n) converges, in which case $\lim(x_n) = \lim(x_{m+n})$

Proof:

($\Leftarrow$) $x_n \to x \Rightarrow \forall \varepsilon > 0$, $\exists N_m \in \mathbb{N} : |x_n - x| < \varepsilon$ $\forall n \geq N_m$

Let $N = N_m \Rightarrow \forall n \geq N_m$ $m + n \geq n \geq N_m = N$

$\Rightarrow |x_{m+n} - x| < \varepsilon$ So, $x_{m+n} \to x$

($\Rightarrow$) $x_{m+n} \to x \Rightarrow \forall \varepsilon > 0$, $\exists N_m \in \mathbb{N} : |x_{m+n} - x| < \varepsilon$ $\forall n \geq N_m$

Let $N_{m+n} = N \Rightarrow \forall n \geq N$ $n' = n - m \geq N - m = N_m$

$\Rightarrow |x_n - x| = |x_{m+n'} - x| < \varepsilon$ So, $x_n \to x$

QED

Subsequences & Convergence

Theorem 3.1: Let $x_n \to x$. Then any subsequence (x_{n_k}) of (x_n) also converges to x.

Proof: Let $\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon \ \forall n \geq N$
As (n_k) is strictly increasing $\exists K \in \mathbb{N}$ such that $n_k \geq N \Rightarrow \forall k \geq K \ \ n_k \geq n_K \geq N$ hence, $|x_{n_k} - x| < \varepsilon$ **QED**

Theorem 3.2: Let (x_n) be a sequence in $\mathbb{R}$. Then, the following are equivalent

1. $x_n \not\to x$
2. $\exists \varepsilon_0 > 0$ such that $\forall k \in \mathbb{N} \ \exists n_k \in \mathbb{N}$ such that $n_k \geq k$ & $|x_{n_k} - x| \geq \varepsilon_0$
3. $\exists \varepsilon_0 > 0$ & $\exists$ a subsequence (x_{n_k}) such that $|x_{n_k} - x| \geq \varepsilon_0 \ \forall k \in \mathbb{N}$

Proof:

1. $\Rightarrow$ 2. As $x_n \not\to x$, $\exists \varepsilon_0 > 0$ such that "$\forall n \geq N \ |x_n - x| < \varepsilon_0$" is false $\Rightarrow \exists n(N) \geq N$ such that $|x_{n(N)} - x| \geq \varepsilon_0$

2. $\Rightarrow$ 3. Pick $\varepsilon_0 > 0$ as in 2.
Pick $k = 1 \Rightarrow$ get $n_1 \geq 1$ & $|x_{n_1} - x| \geq \varepsilon_0$
$k = n_1 + 1 \Rightarrow$ get $n_2 > n_1$ & $|x_{n_2} - x| \geq \varepsilon_0$
$\vdots \qquad \vdots \qquad \vdots$
eventually obtain (x_{n_k})

3. $\Rightarrow$ 1. Suppose that $x_n \to x$, by theorem 3.1 any subsequence $x_{n_k} \to x$, but this contradicts 3., Thus $x_n \not\to x$.

QED

Divergence & Bolzano-Weierstrass

Theorem 3.3: Divergence Criteria

Let (X_n) be a sequence in $\mathbb{R}$ satisfying $\boxed{\text{either}}$

1. has two subsequences (X_{n_k}) & (X_{m_k}) who's limits are not equal.

or

2. unbounded

Bolzano-Weirstrass Theorem:

A bounded sequence of real numbers has a convergent subsequence.

Proof:

$\boxed{\text{Lemma}}$: Let (X_n) be a sequence in $\mathbb{R}$ then there exists a subsequence of (X_n) that is monotone.

→ we say that X_m is a *peak* if $X_m \geq X_k$ $\forall k \geq m$.

<u>case 1:</u> (X_n) has infinitely many peaks.

$\Rightarrow (X_{m_1}, X_{m_2}, \dots)$ with $m_1 < m_2 < \cdots$ (increasing subscript) is a monotone (decreasing) subsequence.

<u>case 2:</u> (X_n) has finitely many peaks.

let $\{X_{m_1}, X_{m_2}, \dots, X_{m_k}\}$ be peaks with $m_1 < m_2 < \dots < m_k$

Take $s_1 = m_k + 1$. As X_{s_1} is not a peak $\exists s_2 : s_2 > s_1$ such that $X_{s_2} > X_{s_1}$. Then, since s_2 is not a peak $\exists s_3 : s_3 > s_2$ such that $X_{s_3} > X_{s_2}$. In this way we can find an increasing subsequence comprised of $(X_{s_1}, X_{s_2}, \dots)$ $\square$

Now, to prove Bolzano-Weierstrass take a monotone subsequence (X_{n_k}) of (X_n), & since (X_n) is bounded, (X_{n_k}) is also bounded. Thus, (X_{n_k}) is convergent by the monotone convergence theorem.

QED

Divergent Sequence Example

$\boxed{\text{Divergence Example}}$ $(\sin(n) : n \in \mathbb{N})$ is divergent.

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = \sin\left(\frac{5\pi}{6}\right)$

Let $I_1 = [\pi/6, 5\pi/6] \Rightarrow \sin(x) > 1/2 \quad \forall x \in I_1$

$|I_1| = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{6} > 2 \Rightarrow \exists n_1 \in \mathbb{N}$ such that $n_1 \in I_1$

Let $I_k = \left[\frac{\pi}{6} + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1)\right] \quad \forall k \geq 2$

$\Rightarrow |I_k| > 2$, so $\exists n_k \in \mathbb{N}$ such that $n_k \in I_k$

We get a subsequence (n_k) such that $(\sin(n_k))$ takes values in $[1/2, 1]$. Similarly, let

$J_k = \left[\frac{7\pi}{6} + 2\pi(k-1), \frac{11\pi}{6} + 2\pi(k-1)\right]$

$\Rightarrow |J_k| > 2$, $\exists m_k \in \mathbb{N}$ such that $m_k \in J_k$

We get a subsequence (m_k) such that $(\sin(m_k))$ takes values in $[-1, -1/2]$

Thus since $(\sin(n_k))$ & $(\sin(m_k))$ cannot converge to the same value $(\sin(n))$ must be divergent.

Subsequential Limits

Let (x_n) be a bounded sequence in $\mathbb{R}$. We say that x is a **subsequential limit** of (x_n) if there exists a subsequence (x_{n_k}) of (x_n) such that $\lim(x_{n_k}) = x$. The set of all subsequential limits of a sequence is often denoted by S.

By considering the $\sup(S)$ & the $\inf(S)$ we can get a more flexible way to show that certain portions of a sequence approach a certain value.

The **limit Superior** of (x_n) is the infimum of the set V of v such that $v < x_n$ for, at most, a finite number of $n \in \mathbb{N}$

↳ denoted by $\limsup(x_n)$ or $\overline{\lim}(x_n)$

Conversely, the **limit Inferior** of (x_n) is the supremum of the set W of w such that $w > x_n$ for, at most, finitely many $n \in \mathbb{N}$

↳ denoted by $\liminf(x_n)$ or $\underline{\lim}(x_n)$

We can think of the limsup as the largest value that the subsequences of (x_n) can keep getting arbitrarily close to infinitely often, & the liminf as the smallest value that the subsequences of (x_n) can keep getting arbitrarily close to infinitely often.

$u_m = \sup(\{x_n : n \geq m\})$

$$\limsup(x_n) = \inf(u_m) = \lim(u_m)$$

$$\liminf(x_n) = \sup(v_m) = \lim(v_m)$$

↳ $v_m = \inf(\{x_n : n \geq m\})$

Analytic Definition of limsup

Theorem 3.6: Let (x_n) be a bounded sequence in $\mathbb{R}$ & let $x^* \in \mathbb{R}$. Then, the following statements are equivalent:

a. $x^* = \limsup(x_n)$

b. If $\varepsilon > 0$, then there are at most finitely many n such that $x^* + \varepsilon < x_n$ but infinitely many m such that $x^* - \varepsilon < x_m$.

c. If $u_m := \sup(\{x_n : n \geq m\})$, then, $x^* = \inf(\{u_m : m \in \mathbb{N}\})$ & $\lim(u_m) = x^*$

d. If S is the set of all subsequential limits of (x_n), then $x^* = \sup(S)$

Proof:

(a. $\Rightarrow$ b.) Let $\varepsilon > 0$. Let x^* be the infimum of V $\exists v \in V$ such that $x^* + \varepsilon > v \geq x^*$ & that there are at most finitely many $n \in \mathbb{N}$ such that $v < x_n$ So, there are finitely many $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$ & $x^* - \varepsilon \notin V$ meaning that there are infinitely many m such that $x^* - \varepsilon < x_m$.

(b. $\Rightarrow$ c.) Let $\varepsilon > 0$. First, for $m_1 < m_2$,
$\{x_n : n \geq m_1\} \supseteq \{x_n : n \geq m_2\}$
$\Rightarrow u_{m_1} \geq u_{m_2}$ so (u_m) is decreasing
$\Rightarrow \inf(u_m) = \lim(u_m)$ As there are finitely many n such that $x^* - \varepsilon < x_n$,
we have: $x^* - \varepsilon < u_m = \sup(\{x_n : n \geq m\})$ $\forall m \in \mathbb{N}$
$\Rightarrow \lim(u_m) \geq x^* - \varepsilon$
Since ε is arbitrary we get $\lim(u_m) = x^*$

Proof continued on the next page.

Proof (cont...)

(c. $\Rightarrow$ d.) Take (x_{n_k}) to be a convergent subsequence since $n_k \geq k$, $u_k = \sup(\{x_n : n \geq k\}) \geq x_{n_k}$
$\Rightarrow x^* = \lim(u_k) \geq \lim(x_{n_k}) \Rightarrow x^* \geq \sup(S)$.
Conversely, we will construct $(x_{n_k}) \to x^*$
As $u_1 = \sup(\{x_n : n \geq 1\})$, choose n_1 such that $u_1 - 1 \leq x_{n_1} \leq u_1$. Inductively choose n_k such that $n_{k+1} > n_k$ & $u_{n_{k+1}} - \frac{1}{k+1} < x_{n_{k+1}} \leq u_{n_{k+1}}$
Take the limit
& using $\lim(u_n) = x^*$ we get $x_{n_k} \to x^* \in S$
hence $x^* \leq \sup(S)$

(d. $\Rightarrow$ a.) Let $\varepsilon > 0$.
Suppose for a contradiction that there exists infinitely many n such that $x^* + \varepsilon < x_n$ If this were the case, then we could select (x_{n_k}) & $\lim(x_{n_k}) \geq x^* + \varepsilon$ ✗
So $x^* + \varepsilon \in V$, hence $\inf(V) = \limsup(x_n) \leq x^* - \varepsilon$
On the other hand, $x^* - \varepsilon < x^* = \sup(S)$ so $\exists (x_{n_k})$ such that $\lim(x_{n_k}) > x^* - \varepsilon$
$\Rightarrow$ There are infinitely many x_n such that $x_n > x^* - \varepsilon \Rightarrow x^* - \varepsilon \notin V$. In fact, $x^* - \varepsilon < v \ \forall v \in V$ because, otherwise $\exists v \leq x^* - \varepsilon \leq x_n$ for infinitely many n which is a contradiction.
So, $x^* - \varepsilon \leq \inf(V) = \limsup(x_n)$.
As ε is arbitrary our work is done.

QED

Theorem: A bounded sequence (X_n) is convergent if & only if $\limsup(X_n) = \liminf(X_n)$

Proof:

Let $s_n = \sup\{X_k : k \geq n\}$ & $i_n = \inf\{X_k : k \geq n\}$

$\limsup(X_n) = \lim(s_n)$ & $\liminf(X_n) = \lim(i_n)$

($\Rightarrow$) (X_n) converges $\Rightarrow \limsup(X_n) = \liminf(X_n) = \lim(X_n)$

Suppose that $X_n \to L$.

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $|X_n - L| < \varepsilon \ \forall n \geq N$

$\Rightarrow i_n \geq L - \varepsilon$ & $s_n \leq L + \varepsilon \quad \forall n \geq N$

$\liminf(X_n) = \lim(i_n) \geq L - \varepsilon$

$\limsup(X_n) = \lim(s_n) \leq L + \varepsilon$

but, $\liminf(X_n) \leq L \leq \limsup(X_n)$

$\Rightarrow \liminf(X_n) = L = \limsup(X_n)$

($\Leftarrow$) $\limsup(X_n) = \liminf(X_n) \Rightarrow (X_n)$ converges

let $\alpha = \liminf(X_n)$ & $\beta = \limsup(X_n)$

suppose for a contradiction that $\alpha = \beta$ but (X_n) does not converge. This means that $\exists \varepsilon_0 > 0$ & a subsequence (X_{n_k}) such that $|X_{n_k} - L| \geq \varepsilon_0 \ \forall k \in \mathbb{N}$ where $L = \alpha = \beta$. Meaning either $X_{n_k} \geq L + \varepsilon_0$ infinitely often <u>or</u>

$X_{n_k} \leq L - \varepsilon_0$ infinitely often

a. $X_{n_k} \geq L + \varepsilon_0 \Rightarrow \beta \geq X_{n_k} \geq L + \varepsilon_0$

$\Rightarrow \limsup(X_n) \geq L + \varepsilon_0$. contradicting $\beta = L$

b. $X_{n_k} \leq L - \varepsilon_0 \Rightarrow \alpha \leq X_{n_k} \leq L - \varepsilon_0$

$\Rightarrow \liminf(X_n) \leq L - \varepsilon_0$ contradicting $\alpha = L$

thus, (X_n) converges.

QED

The Cauchy Criterion

Convergent sequences cluster towards ther limit, meaning the distance between values gets smaller as they approach their limit. Do sequences that cluster like this converge? In more general metric spaces: no, not necessarily. But! in $\mathbb{R}$ the answer is yes!

A sequence (x_n) of real numbers is called a **Cauchy Sequence** if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n \geq m \geq N$

$\boxed{\text{Lemma}}$: Convergent sequences are cauchy.

Let $\lim(x_n) = x$. Then $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $|x - x_n| < \frac{\varepsilon}{2}$. Hence, if $n \geq m \geq N$, then $|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ **QED**

$\boxed{\text{Lemma}}$: Cauchy sequences are bounded.

Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ if $n \geq m \geq N$. Hence, $|x_n - x_N| < \varepsilon$ $\forall n \geq N$, in particular. So, let $M = \max\{|x_1|, \dots, |x_{N-1}|, |x_N| + \varepsilon\}$. Then, $|x_n| \leq M$, $\forall n \in \mathbb{N}$. **QED**

Cauchy Sequences & Convergence

Theorem: A sequence of real numbers is convergent if & only if the sequence is a cauchy sequence.

Proof:

($\Rightarrow$) See the two lemmas on the previous page.

($\Leftarrow$) Let (X_n) be a cauchy sequence in $\mathbb{R}$.
$\Rightarrow (X_n)$ is bounded (by the second lemma on the previous page.) Thus by the Bolzano-Weierstrass theorem, there exists a convergent subsequence of (X_n), say (X_{n_k}).
Let $\lim(X_{n_k}) = X$. Since (X_n) is cauchy, if $\varepsilon > 0$
$\Rightarrow \exists N_1 \in \mathbb{N}$ such that $|X_n - X_m| < \frac{\varepsilon}{2}$ $\forall n \geq m \geq N_1$
Since (X_{n_k}) converges to X $\exists N_2 \in \mathbb{N}$ such that $|X_{n_k} - X| < \frac{\varepsilon}{2}$ $\forall n_k \geq k \geq N_2$. Let $n \geq \max\{N_1, N_2\}$

$$\Rightarrow |X - X_n| = |X - X_N + X_N - X_n|$$
$$\leq |X - X_N| + |X_N - X_n|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\lim(X_n) = X$.

QED

Cauchy Sequence Example

$\boxed{\text{example}}$: let (X_n) be a sequence such that:

$$X_1 := 1 \quad \text{if } n=1$$
$$X_2 := 2 \quad \text{if } n=2$$
$$X_n := \tfrac{1}{2}(X_{n-1} + X_{n-2}) \quad \text{if } n \geq 3$$

The following facts can be proven via induction

i. $1 \leq X_n \leq 2 \quad \forall n \in \mathbb{N}$

ii. $|X_{n+1} - X_n| = \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}$

iii. $X_{2n+1} = 1 + \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^n\right) \quad \forall n \in \mathbb{N}$

So,
$$\begin{aligned}
|X_n - X_m| &= |X_n - X_{n+1} + X_{n+1} - X_{n+2} + X_{n+2} - \cdots + X_{m-1} - X_m| \\
&\leq |X_n - X_{n+1}| + |X_{n+1} - X_{n+2}| + \cdots + |X_{m-1} - X_m| \\
&= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \\
&= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) \\
&= \frac{1}{2^{n-1}}\left(\frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}}\right) = \frac{1}{2^{n-2}}\left(1 - \left(\frac{1}{2}\right)^{m-n}\right) \leq \frac{1}{2^{n-2}} \quad \forall m \geq n
\end{aligned}$$

Thus, let $\varepsilon > 0$ & choose $N \in \mathbb{N}$ such that

$$\frac{4}{\varepsilon} < n < 2^n \quad \forall n \geq N \Rightarrow \frac{1}{2^n} < \frac{\varepsilon}{4} \quad \text{if } n \geq N$$

Hence, if $m \geq n \geq N$ then $|X_n - X_m| \leq \frac{1}{2^{n-2}} < \varepsilon$

Finally,
$$\begin{aligned}
\lim(X_n) &= \lim(X_{2n+1}) \\
&= \lim\left(1 + \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)\right) \\
&= 1 + \frac{2}{3} = 5/3
\end{aligned}$$

Therefore, X_n is a cauchy sequence & $\lim(X_n) = 5/3$

Contractive Sequences

Let (x_n) be a sequence in $\mathbb{R}$. (x_n) is said to be **contractive** if $\exists c \in (0,1)$ such that:

$$|x_{n+2} - x_{n+1}| \leq c\,|x_{n+1} - x_n| \quad \forall n \in \mathbb{N}$$

Theorem: Every contractive sequence is also a cauchy sequence.

Proof:

$$|x_{n+2} - x_{n+1}| \leq c\,|x_{n+1} - x_n| \leq c^2|x_n - x_{n-1}| \leq \cdots \leq c^n|x_2 - x_1|$$

$$\Rightarrow |x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n|$$

$$\leq (c^{m-2} + \cdots + c^{n-1})\,|x_2 - x_1|$$

$$= c^{n-1}\left(\frac{1-c^{m-n}}{1-c}\right)|x_2 - x_1| \leq \frac{c^{n-1}}{1-c}|x_2 - x_1| \quad \forall m \geq n$$

Let $\varepsilon > 0$. Since $\lim(c^n) = \lim(c^{n-1}) = 0$ we have $\lim\left(\frac{c^{n-1}}{1-c}|x_2 - x_1|\right) = 0$. So, $\exists N \in \mathbb{N}$ such that:

$$\frac{c^{n-1}}{1-c}|x_2 - x_1| < \varepsilon \quad \forall n \geq N$$

Thus, $m \geq n \geq N \Rightarrow |x_m - x_n| < \varepsilon$

QED

$\boxed{\text{corallary}}$: If (x_n) is contractive with $c \in (0,1)$ & $\lim(x_n) = x$

Then,

$$|x - x_n| \leq \frac{c^{n-1}}{1-c}|x_2 - x_1| \quad \& \quad |x - x_n| \leq \frac{c}{1-c}|x_n - x_{n+1}$$

Proof: Fix $n \in \mathbb{N}$, then by the proof above we have $|x_m - x_n| \leq \frac{c^{n-1}}{1-c}|x_2 - x_1|$ which implies that $\lim(|x_m - x_n|) = |x - x_n|$

& hence $|x - x_n| < \frac{c^{n-1}}{1-c}|x_2 - x_1|$ Next, observe

$$m \geq n \Rightarrow |x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n|$$

$$\leq (c^{m-n} + \cdots + c)\,|x_{n+1} - x_n| = c\left(\frac{1-c^{m-n}}{1-c}\right)|x_{n+1} - x_n|$$

$$\lim(|x_m - x_n|) \leq \frac{c}{1-c}|x_{n+1} - x_n|$$

QEP

$\boxed{\text{example:}}$ Consider the equation $x^3 - 7x + 2 = 0$.
Let $x_1 \in (0,1)$ be arbitrary & define $x_{n+1} := \frac{1}{7}(x_n^3 + 2)$ $\forall n \in \mathbb{N}$
Then suppose that
$x_k \in (0,1)$ $\forall k \in \mathbb{N}_m$ for some $m \in \mathbb{N}$. Then,

$$x_{m+1} = \frac{1}{7}(x_m^3 + 2) = \frac{1}{7}x_m^3 + \frac{2}{7} \leq \frac{1}{7} + \frac{2}{7} = \frac{3}{7} < 0$$

Furthermore, $x_{m+1} = \frac{1}{7}(x_m^3 + 2) \geq \frac{1}{7}(0+2) = \frac{2}{7} > 0$
So, $x_n \in (0,1)$ $\forall n \in \mathbb{N}$
In fact, $|x_{n+2} - x_{n+1}| = \frac{1}{7}\left|(x_{n+1}^3 + 2) - (x_n^3 + 2)\right|$

$$= \frac{1}{7}\left|x_{n+1}^3 - x_n^3\right|$$

Since $x_n^2 > 1$,
$x_{n+1}^2 > 1$, &
$x_{n+1}x_n > 1$

$$= \frac{1}{7}\left|x_{n+1}^2 + x_{n+1}x_n + x_n^2\right| \cdot \left|x_{n+1} - x_n\right|$$

$$\leq \frac{3}{7}\left|x_{n+1} - x_n\right|$$

$\Rightarrow (x_n)$ is contractive & hence, convergent
Therefore, $x^3 - 7x + 2 = 0$ has a real solution in $[0,1]$ since $x_n \in (0,1)$ $\forall n \in \mathbb{N}$

Theorem: let (X_n) be a bounded sequence in $\mathbb{R}$ & let $x \in \mathbb{R}$ such that every convergent subsequence of (X_n) converges to x. Then (X_n) converges to x.

Proof: Let $M \in \mathbb{R}$ such that $|X_n| \leq M$ $\forall n \in \mathbb{N}$. If (X_n) does not converge to x then $\exists \varepsilon_0 > 0$ & some subsequence (X_{n_k}) such that: $|X_{n_k} - x| \geq \varepsilon_0$ $\forall k \in \mathbb{N}$. However, (X_{n_k}) is bounded since $|X_{n_k}| \leq M$ $\forall k \in \mathbb{N}$. By the Bolzano-Weierstrass theorem (X_{n_k}) has a convergent subsequence, (X_{m_k}), but any convergent subsequence of (X_n) converges to x & so $\exists N \in \mathbb{N}$ such that $|X_{m_k} - x| < \varepsilon_0$ $\forall m_k \geq N$. This condradicts our assumption that (X_n) does not converge to x. Thus, (X_n) must converge to x.

QED

Properly Divergent Sequences

We say that a sequence (X_n) goes to ∞ & write $\lim(X_n) = \infty$ if for every $\alpha \in \mathbb{R}$ there exists a $K(\alpha) \in \mathbb{N}$ such that if $n \geq K(\alpha)$, then $X_n > \alpha$

Similarly, we say that a sequence (X_n) goes to $-\infty$ & write, $\lim(X_n) = -\infty$ if for every $\beta \in \mathbb{R}$ there exists a $K(\beta) \in \mathbb{N}$ such that if $n \geq K(\beta)$, then $X_n < \beta$

We say that (X_n) is **properly divergent** if $\lim(X_n) = \infty$ or $\lim(X_n) = -\infty$

Theorem: A monotone sequence of real numbers is properly divergent if & only if it is unbounded.

a. If (X_n) is unbounded & increasing $\Rightarrow \lim(X_n) = \infty$

b. If (X_n) is unbounded & decreasing $\Rightarrow \lim(X_n) = -\infty$

Proof:

a. Suppose that (X_n) is increasing.

$\Rightarrow$ If (X_n) is bounded, then it converges by the MCT.

$\Rightarrow$ If (X_n) is unbounded then $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}$ such that $M < X_N$. Since (X_n) is increasing $\forall n \geq N$ $M < X_N \leq X_n$. Thus $\lim(X_n) = \infty$

b. Suppose that (X_n) is decreasing

$\Rightarrow$ If (X_n) is bounded, then it converges by the MCT.

$\Rightarrow$ If (X_n) is unbounded, then $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}$ such that $X_N < M$. Since (X_n) is decreasing $\forall n \geq N$ $X_n \leq X_N < M$. Thus $\lim(X_n) = -\infty$

QED

Theorem: Let (x_n) & (y_n) be sequences in $\mathbb{R}$ such that $x_n \leq y_n$ $\forall n \in \mathbb{N}$.

a. $\lim(x_n) = \infty \Rightarrow \lim(y_n) = \infty$

b. $\lim(y_n) = -\infty \Rightarrow \lim(x_n) = -\infty$

Proof:

a. If $\lim(x_n) = \infty$ then, (x_n) is unbounded & increasing
$\Rightarrow \forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ such that $M < x_n \leq y_n$ $\forall n \geq N$
$\Rightarrow M < y_n$ $\forall M \in \mathbb{R}$ & $\forall n \geq N \Rightarrow \lim(y_n) = \infty$

b. If $\lim(y_n) = -\infty$ then, (y_n) is unbounded & decreasing
$\Rightarrow \forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ such that $x_n \leq y_n < M$ $\forall n \geq N$
$\Rightarrow M > x_n$ $\forall M \in \mathbb{R}$ & $\forall n \geq N \Rightarrow \lim(x_n) = -\infty$

QED

Theorem: Let (x_n) & (y_n) be positive sequences in $\mathbb{R}$ such that $\lim\left(\frac{x_n}{y_n}\right) = L$ for some $L > 0 \in \mathbb{R}$. Then,
$\lim(x_n) = \infty \Leftrightarrow \lim(y_n) = \infty$

Proof: Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that
$n \geq N \Rightarrow \left|\frac{x_n}{y_n} - L\right| < \varepsilon \Rightarrow -\varepsilon < \frac{x_n}{y_n} - L < \varepsilon$

$$\Rightarrow L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon$$

$$\Rightarrow y_n(L - \varepsilon) < x_n < y_n(L + \varepsilon)$$

set ε such that $L - \varepsilon > 0$. Then, the desired result follows from the above theorem applied to $(y_n(L-\varepsilon))$ & $(y_n(L+\varepsilon))$

QED

Introduction to Infinite Series

If (X_n) is a sequence in $\mathbb{R}$, then the **infinite series** generated by (X_n) is the sequence (S_n) defined by

$$S_n := \sum_{i=1}^{n} X_i \quad \forall n \in \mathbb{N} \qquad S_{n+1} = S_n + X_{n+1}$$

Theorem: If $\sum X_n$ converges, then $\lim(X_n) = 0$

Proof: If $\sum X_n$ converges to $L \in \mathbb{R}$, then

$$\lim(X_n) = \lim(X_{n+1}) = \lim(S_{n+1} - S_n) = \lim(S_{n+1}) - \lim(S_n) = 0$$

QED

Note:

The converse does not always hold in $\mathbb{R}$. However, it does hold for non-Archimedean fields such as $\mathbb{Q}_p$ where p is a prime number.

Theorem: The series $\sum X_n$ converges $\iff$ $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $|S_m - S_n| = \left| \sum_{i=n+1}^{m} X_i \right| < \varepsilon$ when $m \geq n \geq N$

Proof: (S_n) converges $\iff$ (S_n) is cauchy

QED

Theorem: Let (X_n) be a non-negative sequence. Then $\sum X_n$ converges $\iff$ $(S_n) = \left(\sum_{i=1}^{n} X_i\right)$ is bounded. Moreover, if (S_n) converges, then $\sum_{n=1}^{\infty} X_n = \lim(S_n) = \sup(\{S_n : n \in \mathbb{N}\})$

Proof: If $X_n \geq 0 \ \forall n \in \mathbb{N}$, then $S_{n+1} = S_n + X_{n+1} \geq S_n \ \forall n \in \mathbb{N}$ $\Rightarrow (S_n)$ is increasing, thus the result follows from the monotone convergence theorem.

QED

$\boxed{\text{Example 1}}$: Let (r^n) be a sequence in $\mathbb{R}$ such that $|r|<1$.

$\Rightarrow (S_n) = \left(\sum_{n=0}^{\infty} r^n\right)$ is called a **geometric series**

$\boxed{\text{Note: } S_n(1-r) = 1 - r^{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}}$

$\left| S_n - \frac{1}{1-r} \right| = \frac{|r|^{n+1}}{|1-r|}$ Since we know that $0 \leq |r| < 1$ it follows that $\lim(|r|^{n+1}) = 0$. Thus, $\lim(S_n) = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

$\boxed{\text{Example 2}}$ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ $\quad X_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\Rightarrow$ The partial sums are $S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

$S_n = 1 - \frac{1}{n+1}$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim(S_n)$$
$$= \lim\left(1 - \frac{1}{n+1}\right) = 1$$

$\boxed{\text{Example 3}}$: Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (This is called the Basel Problem)

Since each $\frac{1}{n^2}$ is positive, the partial sums give an increasing sequence. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \iff \text{The partial sums are bounded}$$

Observe that

$$S_1 = 1 \leq 1$$

$$S_3 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 \leq 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{2}$$

$$S_7 = S_3 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{7}\right)^2$$

$\left(\begin{array}{l}(S_n) \text{ is increasing} \\ \text{since } \frac{1}{n^2} > 0 \ \forall n \in \mathbb{N}\end{array}\right)$

$$\leq \frac{3}{2} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{2} + \frac{1}{4}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2$$

$$S_{2^n-1} = S_{2^{n-1}-1} + \left(\frac{1}{2^{n-1}}\right)^2 + \cdots + \left(\frac{1}{2^n-1}\right)^2 \leq S_{2^{n-1}-1} + \overbrace{\left(\frac{1}{2^{n-1}}\right)^2 + \cdots + \left(\frac{1}{2^{n-1}}\right)^2}^{2^{n-1} \text{ terms}}$$

$$\leq 1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{n-2} + \left(\frac{1}{2}\right)^{n-1}$$

Can be shown via induction

$$S_{2^{n-1}-1} \leq 1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{n-2}$$

$$= 2\left(1 - \left(\frac{1}{2}\right)^n\right) \leq 2$$

Thus, the (2^n-1)-partial sums are bounded, & since $\lim(2^n-1) = \infty$ this implies that (S_n) is bounded. Thus, (S_n) is convergent by the MCT.

↳ To prove this fully on an assignment we would have to prove that $\lim(2^n-1) = \infty$ by showing that it is unbounded & increasing!

QED

$\boxed{\text{Example 4}}$ Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

$S_{2n} = (1-\frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{2n-1} - \frac{1}{2n})$

$\Rightarrow (S_{2n})$ is increasing.

$S_{2n+1} = 1 - (\frac{1}{2} - \frac{1}{3}) - \cdots - (\frac{1}{2n} - \frac{1}{2n+1})$

$\Rightarrow (S_{2n+1})$ is decreasing. However,

$0 < S_{2n} + \frac{1}{2n+1} = S_{2n+1} \leq 1 \quad \forall n \in \mathbb{N}$

$\Rightarrow (S_{2n})$ is bounded above &
(S_{2n+1}) is bounded below.

$\Rightarrow \lim(S_{2n})$ & $\lim(S_{2n+1})$ both exist meaning

$\lim(S_{2n+1}) = \lim(S_{2n} + \frac{1}{2n+1}) = \lim(S_{2n})$

implies that $\lim(S_n)$ exists as well!

Theorem: Let (x_n) & (y_n) be sequences in $\mathbb{R}$ such that $0 \leq x_n \leq y_n$ eventually. Then:

(a.) $\sum y_n$ converges $\Rightarrow \sum x_n$ converges.

(b.) $\sum x_n$ diverges $\Rightarrow \sum y_n$ diverges.

Proof:

(a.) $\forall \varepsilon > 0$ $\sum y_n$ converges $\Rightarrow \exists N \in \mathbb{N}$ such that $\sum_{i=n+1}^{m} y_i < \varepsilon$ & $0 \leq x_n \leq y_n$ whenever $m \geq n \geq N$

Hence, $0 \leq \sum_{i=n+1}^{m} x_i \leq \sum_{i=n+1}^{m} y_i < \varepsilon$ $\forall m \geq n \geq N$

$\therefore (\sum x_n)$ is cauchy $\Rightarrow (\sum x_n)$ is convergent

(b.) $\forall M > 0$ $\sum x_n$ diverges $\Rightarrow \exists N \in \mathbb{N}$ such that $\sum_{i=n+1}^{m} x_i > M$ & $0 \leq x_n \leq y_n$ whenever $m \geq n \geq N$

Hence $\sum_{i=n+1}^{m} y_i \geq \sum_{i=n+1}^{m} x_i > M > 0$ $\forall m \geq n \geq N$

$\therefore (\sum y_n)$ is not cauchy $\Rightarrow (\sum y_n)$ is divergent

QED

Theorem: Let (x_n) & (y_n) be positive sequences in $\mathbb{R}$ such that $\lim\left(\frac{x_n}{y_n}\right) = r \in \mathbb{R}$ Then,

(a.) if $r \neq 0$ then,
$(\sum x_n)$ converges $\Leftrightarrow (\sum y_n)$ converges.

(b.) if $r = 0$ then, $(\sum y_n)$ converges $\Rightarrow (\sum x_n)$ converges.

$\boxed{\text{Example 5}}$ consider the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right)$$

$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \boxed{1 + k\left(\frac{1}{2}\right)}$$

This is unbounded, & since it is (strictly) less than the harmonic series, the harmonic series diverges to ∞

Theorem: Cauchy Condensation Test

Let (a_n) be a non-negative decreasing sequence in $\mathbb{R}$. Then, the series $\sum a_n$ converges if & only if the series

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots \text{ converges}$$

Proof:

($\Rightarrow$) Assume that $\sum 2^n a_{2^n}$ converges.

$\Rightarrow$ the partial sums $t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$ are bounded; that is $\exists M > 0$ such that $t_k \leq M \ \forall k \in \mathbb{N}$

$\boxed{\text{NTS: } S_m \text{ is bounded}}$

Since $a_n \geq 0 \ \forall n \in \mathbb{N}$, the partial sums $S_m = a_1 + a_2 + a_3 + \cdots + a_m$ are increasing.

$\forall m \in \mathbb{N} \ \exists k$ such that $m \leq 2^{k+1} - 1 \Rightarrow S_m \leq S_{2^{k+1}-1}$.

$$\begin{aligned} S_{2^{k+1}-1} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + (a_{2^k} + \cdots + a_{2^k}) \\ &= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k \end{aligned}$$

$\therefore S_m \leq t_k \leq M$ meaning (S_m) is bounded.

($\Leftarrow$) Assume that $\sum 2^n a_n$ diverges.

$\Rightarrow$ the partial sums t_k are unbounded; that is, $\forall M > 0 \ \exists k \in \mathbb{N}$ such that $|t_k| > M$

Chapter 4: Limits

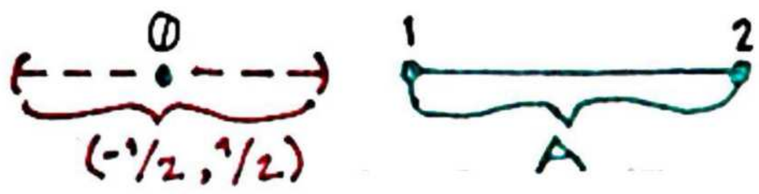
(4.1: Limits of Functions)

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **Cluster Point** of A if $\forall \delta > 0$, $\exists x \in A$ such that $x \neq c$ & $|x-c| < \delta$.

The idea here is that c should <u>not</u> be isolated.

↳ <u>ex:</u> $A = \{0\} \cup [1,2]$, then 0 is <u>not</u> a cluster point of A

set $\delta = 1/2$ & observe that $A \cap (-1/2, 1/2) = \{0\}$



Furthermore, c does <u>not</u> need to belong to A.

> <u>Note:</u> most other text books will refer to a cluster point as a <u>limit point</u>.

Theorem 4.1: $c \in \mathbb{R}$ is a cluster point of $A \subseteq \mathbb{R}$ $\iff$ $\exists (a_n) \in A$ such that $\lim(a_n) = c$ & $a_n \neq c$ $\forall n \in \mathbb{N}$

Proof:

($\Rightarrow$): $\forall n \in \mathbb{N}$, $\exists x_n \in (A \setminus \{c\}) \cap (c - \frac{1}{n}, c + \frac{1}{n})$ since c is a cluster point. Hence, $x_n \neq c$ $\forall n \in \mathbb{N}$, & $|x_n - c| < \frac{1}{n}$ $\forall n \in \mathbb{N}$. Thus $\lim(x_n) = c$ by the definition of a limit

($\Leftarrow$): Let $\delta > 0$, then $\exists N \in \mathbb{N}$ such that: $|a_n - c| < \delta$ $\forall n \geq N$. In particular, $a_N \neq c$ & $a_N \in A \cap (c-\delta, c+\delta)$ meaning c is a cluster point of A.

QED

The set of cluster points of any finite set is $\varnothing$

The set of cluster points of $A = (0,1)$ is $[0,1]$

$\mathbb{N}$ has no cluster points.

The only cluster point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is 0

The set of cluster points of $A = [0,1]$ is $[0,1]$

Let $A \subseteq \mathbb{R}$ & let c be a cluster point of A. We say that a real number L is a limit of $f: A \to \mathbb{R}$ at c if given any $\varepsilon > 0$, $\exists \delta > 0$ such that if $x \in A$ & $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

Theorem 4.2: If $f: A \to \mathbb{R}$ & c is a cluster point of A, then f can have at most one limit at c.

<u>*Proof:*</u> Suppose that both L & L' are limits of f at c. Then, if $\varepsilon > 0$, $\exists \delta_1 > 0, \delta_2 > 0$ such that if $x \in A$ & $\begin{cases} 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2} \\ 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L'| < \frac{\varepsilon}{2} \end{cases}$ Hence, if $x \notin A$ & $0 < |x - c| < \min\{\delta_1, \delta_2\}$, then $|L - L'| \leq |L - f(x)| + |f(x) - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow$ Since $\varepsilon > 0$ is arbitrary, we must have $L = L'$

QED

The above theorem tells us that if a limit of f at c exists, then it is unique. If no such limit exists, we say that f diverges at c. Otherwise, f converges at c & we write $L = \lim_{x \to c}(f(x))$ if L is the limit of f at c.

Theorem 4.3: Let $f: A \to \mathbb{R}$ & let $c \in \mathbb{R}$ be a cluster point of A. Then, the following are equivalent.

(a.) $\lim_{x \to c} (f(x)) = L$.

(b.) Given any $\varepsilon > 0$, $\exists \delta > 0$ such that if $x \in V_\delta(c) \cap A$ then $f(x) \in V_\varepsilon(L)$.

$\boxed{\text{Examples}}$:

(1.) $\lim_{x \to c} (b) = b$ given any $b, c \in \mathbb{R}$.
Let $\varepsilon > 0$. Then for any $\delta > 0$, it follows that if $0 < |x - c| < \delta \Rightarrow |f(x) - b| = |b - b| = 0 < \varepsilon$. If we set $f: \mathbb{R} \to \mathbb{R}$ via $x \mapsto b$ $\forall x \in \mathbb{R}$

(2.) $\lim_{x \to c} (x) = c$
Let $\varepsilon > 0$, then set $\delta = \varepsilon$ which yields $0 < |x - c| < \varepsilon$

(3.) $\lim_{x \to c} (x^2) = c^2$
Let $\varepsilon > 0$ & observe that $|x^2 - c^2| = |x - c| \cdot |x + c|$
So, if $|x - c| < M$ for some fixed $\leq |x - c|(|x| + |c|)$
$M > 0$. Then, $x \in (c - M, c + M) \Rightarrow |x| < |c| + M$. Hence,
If $0 < |x - c| < \min\left\{M, \frac{\varepsilon}{2|c| + M}\right\} \Rightarrow |x^2 - c^2| \leq |x - c| \cdot (|x| + |c|)$

$$< \left(\frac{\varepsilon}{2|c| + M}\right)\left((|c| + M) + |c|\right)$$

$$= \varepsilon$$

④ $\lim_{x \to c} \left(\frac{1}{x}\right) = \frac{1}{c}$ (if & only if $c \neq 0$)

WLOG assume $c > 0$ & let $\varepsilon > 0$

$$\Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{xc}\right| = \frac{1}{c} \cdot \frac{|c-x|}{|x|}$$

Furthermore, if we assume $x > 0$

$$\Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{1}{cx} \cdot |c-x| \leq \frac{1}{c} \cdot \frac{2}{c} \cdot |c-x|$$

$\hookrightarrow \because \frac{c}{2} < x \Leftrightarrow \frac{1}{x} < \frac{2}{c}$

Thus, if $0 < |x-c| < \min\left\{\frac{c}{2}, \frac{c^2\varepsilon}{2}\right\}$

$$\Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right| \leq \frac{2}{c^2} \cdot |c-x| \leq \frac{2}{c^2} \cdot \frac{c^2}{2} \cdot \varepsilon = \varepsilon$$

⑤ $\lim_{x \to 2} \left(\frac{x^3-4}{x^2+1}\right) = \frac{4}{5}$ Let $\varepsilon > 0$

$$\Rightarrow \left|\frac{x^3-4}{x^2+1} - \frac{4}{5}\right| = \left|\frac{5x^3-4x^2-24}{5(x^2+1)}\right| = \frac{|5x^2+6x+12|}{5(x^2+1)} \cdot |x-2|$$

$\because \nexists$ numerator,
& $x^2+1 \geq 1$ in the denominator $\Rightarrow \leq \left(\frac{5|x|^2+6|x|+12}{5}\right) \cdot |x-2|$

So, if we further assume $1 < x < 3$ then,

$$\Rightarrow \left|\frac{x^3-4}{x^2+1} - \frac{4}{5}\right| \leq \left(\frac{5 \times 3^2 + 6 \times 3 + 12}{5}\right)|x-2| = 15|x-2|$$

Hence, if $0 < |x-2| < \min\left\{1, \frac{\varepsilon}{15}\right\}$ then,

$$\left|\frac{x^3-4}{x^2+1} - \frac{4}{5}\right| < 15 \cdot \frac{\varepsilon}{15} < \varepsilon$$

Theorem 4.4: Let $f: A \to \mathbb{R}$ & c be a cluster point of A, then the following are equivalent:

a. $\lim_{x \to c}(f(x)) = L$

b. For all sequences (x_n) in A that converge to c such that $x_n \neq c$ $\forall n \in \mathbb{N}$, we have $\lim(f(x_n)) = L$.

Theorem 4.5: Let $f: A \to \mathbb{R}$ & c be a cluster point of A, then the following are true.

1. If $L \in \mathbb{R}$, then f does <u>NOT</u> have a limit L at c if & only if there exists a sequence (x_n) in A with $x_n \neq c$ $\forall n \in \mathbb{N}$ such that $\lim(x_n) = c$, but $\lim(f(x_n)) \neq L$.

2. f does <u>NOT</u> have a limit at c if & only if there exists a sequence (x_n) in A with $x_n \neq c$ $\forall n \in \mathbb{N}$ such that $\lim(x_n) = c$ but $(f(x_n))$ diverges.

Note: Theorem 4.4 allows us to leverage facts about sequential limits to obtain information about the limits of functions. For example, $\lim_{x \to c}(x^2) = c^2$ because for any sequence (x_n) satisfying $\lim(x_n) = c$ we have $\lim(x_n^2) = c^2$. Hence, if $x_n \neq c$ $\forall n \in \mathbb{N}$ then theorem 4.4 tells us that $\lim_{x \to c}(x^2) = c^2$.

4.2 Limit Theorems

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ & let $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if $\exists\, \delta > 0$ & some $M > 0$ such that $|f(x)| \leq M$ whenever $x \in A \cap V_\delta(c)$

Theorem 4.6: If $A \subseteq \mathbb{R}$ & $f: A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.

<u>*Proof:*</u> If $L = \lim_{x \to c}(f(x))$, then $\forall \varepsilon > 0$, $\exists\, \delta$ such that $x \in A \cap V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(L)$ meaning that $|f(x)| < |L| + \varepsilon$, $\forall x \in A \cap V_\delta(c)$. **QED**

Chapter 5: Continuity

Continuous Functions

5.1 Introduction to Continuity

The "pencil test" is for children we need a more rigorous definition of continuity.

Let $A \subseteq \mathbb{R}$, $c \in A$, & $f: A \to \mathbb{R}$. f is **continuous** at c if given any $\varepsilon > 0$, $\exists \delta > 0$ such that $x \in A \cap V_\delta(c)$ implies that $f(x) \in V_\varepsilon(f(c))$

When evaluating continuity it is important to note that c must be in A which need not be the case for c to be a cluster point of A. If c is also a cluster point of A, then:

$$f \text{ is continuous at } c \iff f(c) = \lim_{x \to c}(f(x))$$

If c is <u>NOT</u> a cluster point of A, then c is an isolated point in A. That is, $\exists \delta > 0$ such that $A \cap V_\delta(c) = \{c\}$. Hence, $f(A \cap V_\delta(c)) = \{f(c)\} \subseteq V_\varepsilon(f(c))$ $\forall \varepsilon > 0$. This means that f is trivially continious at any isolated points in A. However, as mentioned above, $f(x)$ is <u>NOT</u> continuous at c if $c \notin A$.

$\boxed{\text{Theorem}}$: $f: A \to \mathbb{R}$ is continuous at $c \in A$ $\iff$ $\forall$ sequences (x_n) in A converging to c we have $(f(x_n))$ converges to $f(c)$

Note: If $c \in A$ is isolated, then the only sequences in A that converge to c are eventually constant.

$\boxed{\text{Theorem}}$: $f: A \to \mathbb{R}$ is discontinuous at $c \in A$ $\iff$ $\exists$ a sequence (x_n) in A such that $\lim(x_n) = c$ but $\lim(f(x_n)) \neq f(c)$

5.1 Introduction to Continuity

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$. If $B \subseteq A$ then we say f is continuous on B if & only if f is continuous at every point in B.

Examples:

1. All constant functions are continuous on $\mathbb{R}$.
2. Polynomials are continuous on $\mathbb{R}$
3. If $r = \frac{p}{q}$ (where p & q are polynomials) then r is continuous on $\mathbb{R} \setminus \{x \in \mathbb{R} : q(x) = 0\}$

$\boxed{\text{Thomae's Function}}$ Let $f: (0, \infty) \to \mathbb{R}$ via

$$x \longmapsto \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ where } m, n \in \mathbb{N} \text{ \& } (m,n) = 1. \end{cases}$$

↳ For example: $f(\pi) = f(e) = f(\sqrt{2}) = 0$

but, $f(5) = 1$, $f(\frac{4}{17}) = f(\frac{16}{17}) = \frac{1}{17}$, $f(\frac{6}{9}) = \frac{1}{3}$

$\Rightarrow$ f is continuous on $(0, \infty) \cap (\mathbb{R} \setminus \mathbb{Q})$, but f is discontinuous on $(0, \infty) \cap \mathbb{Q}$

Proof:

Let $c = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ with no common prime factors. Then, $f(c) = \frac{1}{n}$ & since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, $\exists$ a sequence (x_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $\lim(x_n) = c$

Furthermore, if we assume $x_n > 0$ $\forall n \in \mathbb{N}$, so $\lim(\underbrace{f(x_n)}_{0}) = 0 \neq \frac{1}{n} = f(c)$.
Hence, f is discontinuous at c.

But if $c \in (0, \infty) \cap (\mathbb{R} \setminus \mathbb{Q})$ & $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Since $c \notin \mathbb{Q}$, it follows that for each $n \in \mathbb{N}_N$, there is a unique $m_n \in \mathbb{N}$ such that $\frac{m_n - 1}{n} < c < \frac{m_n}{n}$. So let $0 < \delta < \min\{|\frac{m_n - 1}{n} - c| \cup |\frac{m_n}{n} - c|\}$ $(n \in \mathbb{N}_N)$

Then, $x \in V_\delta(c) \cap \mathbb{Q} \Rightarrow x = \frac{k}{\ell}$ with $(k, \ell) = 1$ & $\ell > N$

Thus, $\underbrace{x \in V_\delta(c)}_{x > 0 \text{ automatically!}} \Rightarrow f(x) < \frac{1}{N} < \varepsilon \Rightarrow f(x) \in V_\varepsilon(\underbrace{f(c)}_{0})$

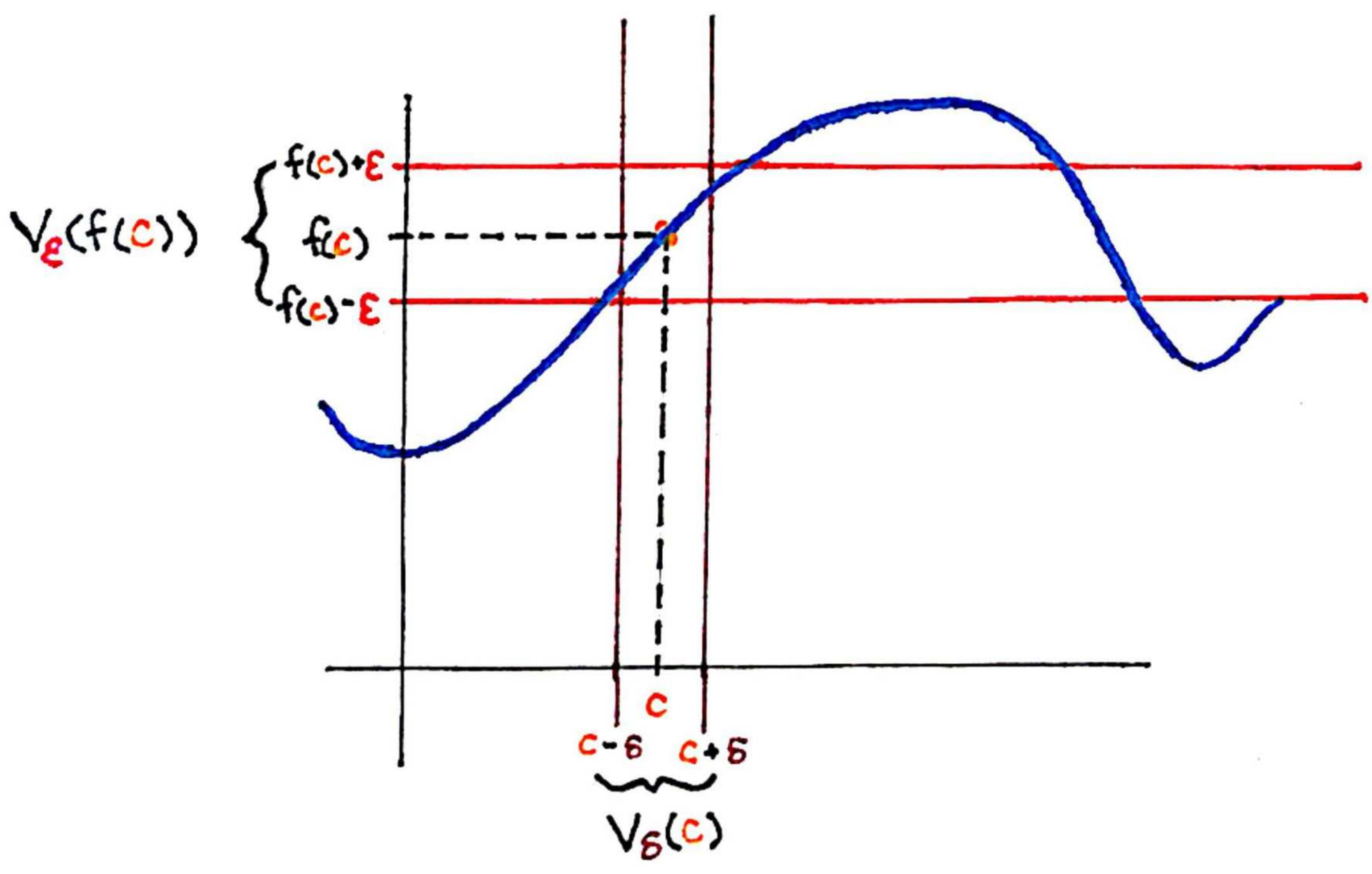
$\therefore$ f is continuous at c

<u>Note</u>: If $c \notin A$, but c is a cluster point of A & if $L = \lim_{x \to c}(f(x))$ exists, then $f: A \to \mathbb{R}$ can be *extended* as follows:

$$F: A \cup \{c\} \to \mathbb{R} \quad \text{via} \quad x \mapsto \begin{cases} f(x) & \text{if } x \in A \\ L & \text{if } x = c \end{cases}$$

The benefit to this is that F is continuous at c. <u>However</u>, if $\lim_{x \to c}(f(x))$ does not exist then we cannot construct an extension to $A \cup \{c\}$ that is continuous at c.

Visualization of continuity



5.2 Combinations of Continuous Functions

Theorem: Combinations of Continuous Functions

Let $A \subseteq \mathbb{R}$, $b \in \mathbb{R}$ & let $f, g, h : A \to \mathbb{R}$ such that $0 \notin h(A)$. Then, if f, g, h are continuous at $c \in A$ then the following are also continuous at c as well

1. $f+g$,
2. $f-g$,
3. fg,
4. bf,
5. $\frac{f}{h}$

Proof:

Recall: f is continious at c if & onlu if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in A$, $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

1&2: NTS: $f \pm g$ is continuous at c

Let $\varepsilon > 0$, then since f & g are continuous at c

$$\Rightarrow \begin{cases} \exists \delta_1 \text{ such that } 0 < |x-c| < \delta_1 \Rightarrow |f(x) - f(c)| < \varepsilon/2 \\ \exists \delta_2 \text{ such that } 0 < |x-c| < \delta_2 \Rightarrow |g(x) - g(c)| < \varepsilon/2 \end{cases}$$

Let $\delta = \min\{\delta_1, \delta_2\}$

$$\begin{aligned} |x-c| < \delta \Rightarrow |(f \pm g)(x) - (f \pm g)(x)| &= |(f(x) - f(c)) \pm (g(x) - g(c))| \\ &\leq |f(x) - f(c)| + |g(x) - g(c)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

3&4: NTS: fg is continuous at c

$$\begin{aligned} |f(x)g(x) - f(c)g(c)| &= |f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)| \\ &\leq |g(x)||f(x) - f(c)| + |f(c)||g(x) - g(c)| \end{aligned}$$

then, since g is continuous at c $\exists \delta_3 > 0$ such that $|x-c| < \delta_3 \Rightarrow |g(x)| \leq |g(c)| + |g(x) - g(c)| < |g(c)| + 1$

Let $\varepsilon > 0$, choose $\delta_4 > 0$ with $|x-c| < \delta_4$ implying $|f(x) - f(c)| < \dfrac{\varepsilon}{2(|g(c)|+1)}$ & choose $\delta_5 > 0$ with $|x-c| < \delta_5$ implying $|g(x) - g(c)| < \dfrac{\varepsilon}{2(\max\{1, f(c)\})}$

Let $\delta^* = \min\{\delta_3, \delta_4, \delta_5\}$

$$|x-c| < \delta^* \Rightarrow |f(x)g(x) - f(c)g(c)|$$

$$\leq (|g(c)|+1)\frac{\varepsilon}{2(|g(c)|+1)} + |f(c)|\frac{\varepsilon}{2(\max\{1, f(c)\})} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(continued on the next page...)

Proof (cont...)

5: NTS: f/h is continuous at c

Since h is continuous at c & $h(c) \neq 0$ $\exists \delta_6 > 0$ such that $|x-c| < \delta_6 \Rightarrow |h(x)-h(c)| < |h(c)|/2$

Therefore, for $|x-c| < \delta_6$

$$|h(x)| \geq |h(c)| - |h(x)-h(c)| > \frac{|h(c)|}{2}$$

$$\Rightarrow \left|\frac{1}{h(x)}\right| \leq \frac{2}{|h(c)|}$$ which allows us to write

$$\left|\frac{f(x)}{h(x)} - \frac{f(c)}{h(c)}\right| = \left|\frac{f(x)h(c) - f(c)h(x)}{h(x)h(c)}\right|$$

$$\leq \frac{1}{|h(c)|}\left|\frac{f(x)-f(c)}{h(x)}\right| + \frac{|f(c)|}{|h(c)|}\left|\frac{h(x)-h(c)}{h(x)}\right|$$

$$\leq \frac{2}{|h(c)|}|f(x)-f(c)| + \frac{2|f(c)|}{|h(c)^2|}|h(x)-h(c)|$$

Let $\varepsilon > 0$ & choose $\delta_7 > 0$ so that

$$|x-c| < \delta_7 \Rightarrow |f(x)-f(c)| < \varepsilon \frac{|h(c)|}{4}$$

& choose $\delta_8 > 0$ so that

$$|x-c| < \delta_8 \Rightarrow |h(x)-h(c)| < \varepsilon \frac{|h(c)|^2}{4(\max\{1, |f(c)|\})}$$

Let $\delta^{**} = \min\{\delta_6, \delta_7, \delta_8\}$

$$\Rightarrow \left|\frac{f(x)}{h(x)} - \frac{f(c)}{h(c)}\right| \leq \frac{2}{|h(c)|}\varepsilon\frac{|h(c)|}{4} + \frac{2|f(c)|}{|h(c)|^2}\varepsilon\frac{|h(c)|^2}{4(\max\{1,|f(c)|\})}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\blacksquare$

Continuous Functions On Intervals

5.3 Continuous Functions on Intervals

$f: A \to \mathbb{R}$ is **bounded on A** if $\exists M > 0$ such that $|f(x)| \leq M$ $\forall x \in A$. Otherwise, f is **unbounded on A**.

$\boxed{\text{Theorem}}$: Let $I = [a,b]$ & let $f: I \to \mathbb{R}$ be continuous, then f is bounded on I.

Proof: Suppose, for a contradiction, that f is unbounded on I. Then, $\forall n \in \mathbb{N}$, $\exists x_n \in I$ such that $|f(x_n)| > n$. But, (x_n) is bounded since I is bounded. So, there must be some subsequence (x_{n_k}) of (x_n) that converges to some $x \in \mathbb{R}$ (By the Bolzano-Wierstrass Theorem). However, $x \in I$ since $x_n \in I$ $\forall n \in \mathbb{N}$. But, the continuity of f at x implies that $(f(x_{n_k}))$ must converge to $f(x)$. Therefore, $(f(x_{n_k}))$ must be bounded, which contradicts our initial assumption since:

$$|f(x_{n_k})| > n_k \geq k \quad \forall k \in \mathbb{N}$$

$\therefore$ f must be bounded on I ∎

Note that each assumption in the previous theorem is necessary in order for it to hold.

1. I must closed both above & below.
2. $f: I \to \mathbb{R}$ must be continuous.

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ we say that f has an **absolute maximum on A** if $\exists x \in A$ such that $f(x) \geq f(y)$ $\forall y \in A$.

Location of Roots Theorem: Let $I=[a,b]$ & let $f: I \to \mathbb{R}$ be continuous on I. If $f(a)<0<f(b)$ or $f(a)>0>f(b)$ then there exists a number $c \in (a,b)$ such that $f(c)=0$.

Proof: Assume that $f(a)<0<f(b)$

Let $I_1=[a_1,b_1]$ where $a_1:=a$ & $b_1:=b$ let p_1 be the midpoint of I_1 meaning $p_1:=\frac{1}{2}(a_1+b_1)$

If $c=p_1$ we are done.

Otherwise, either $f(p_1)>0$ or $f(p_1)<0$

If $f(p_1)>0 \Rightarrow$ set $a_2:=a_1$ & $b_2:=p_1$

If $f(p_1)<0 \Rightarrow$ set $a_2:=p_1$ & $b_2:=b_1$

In either case, $I_2:=[a_2,b_2] \subset I_1$ & $f(a_2)<0<f(b_2)$

We can repeat this process to obtain nested intervals $I_1, I_2, \dots, I_k$. Set $p_k:=\frac{1}{2}(a_k+b_k)$

If $f(p_k)=0$ then $c=p_k$ & we are done.

Otherwise, either $f(p_k)>0$ or $f(p_k)<0$

If $f(p_k)>0 \Rightarrow$ set $a_{k+1}:=a_k$ & $b_{k+1}:=p_k$

If $f(p_k)<0 \Rightarrow$ set $a_{k+1}:=p_k$ & $b_{k+1}:=b_k$

If this process terminates then we are done; If this process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n:=[a_n,b_n]$ such that $\forall n \in \mathbb{N}$ we have $f(a_n)<0$ & $f(b_n)>0$. Furthermore, since the intervals are all bisections the length of I_n is equal to $a_n-b_n=(b-a)/2^{n-1}$. By the nested interval property $\exists c \in I_n \ \forall n \in \mathbb{N}$ & since f is continuous $a_n \le c \le b_n$ for all $n \in \mathbb{N}$ & $\lim(a_n-b_n)=0$ we have $\lim(f(a_n))=\lim(f(c))=\lim(f(b_n))$

$$f(c)=\lim(f(a_n)) \le 0 \ \& \ f(c)=\lim(f(b_n)) \ge 0$$

This implies that $f(c)=0$ as desired. ∎

We use the BWT & A sequence of supremums to show that $f(c)=0$.

Alternate Proof: Let $W := \{x \in I : f(x) < 0\}$ & $w := \sup(W)$

$\boxed{\text{NTS: } f(w) = 0}$ $W \subseteq [a,b] \Rightarrow a \leq w \leq b$ which implies that $w \in I$

↳ We can do this by showing $f(w) \leq 0$ & $f(w) \geq 0$

<u>Step 1:</u> Show that $f(w) \leq 0$

Assume for a contradiction that $f(w) > 0$.

Since f is continuous on I, $\exists \delta > 0$ such that

$|x - w| < \delta \Rightarrow |f(x) - f(w)| < f(w)$

$\Rightarrow f(x) > f(w) - f(w) = 0 \Rightarrow f(x) > 0 \ \forall x \in (w-\delta, w+\delta) \cap I$

$\Rightarrow \exists x' \in W$ such that $w - \delta < x' \leq w \Rightarrow f(x') < 0$,

but this contradicts $f(x') > 0$.

$\therefore f(w) \leq 0$

<u>Step 2:</u> show that $f(w) \geq 0$

Assume for a contradiction that $f(w) < 0$.

Since f is continuous on I, $\exists \delta$ such that

$|x - w| < \delta \Rightarrow |f(x) - f(w)| < |f(w)|$

$\Rightarrow f(x) < f(w) + |f(w)| = 0 \Rightarrow f(x) < 0 \ \forall x \in (w-\delta, w+\delta) \cap I$

$\Rightarrow \exists x' \in W$ such that $\underbrace{w < x'}$ $\leq w + \delta \Rightarrow f(x') < 0$,

but this contradicts $\sup(W) = w$.

$\therefore f(w) \geq 0$

$f(w) \leq 0$ & $f(w) \geq 0 \Rightarrow f(w) = 0$. ∎

Extreme Value Theorem

5.3 Continuous Functions on Intervals

$\boxed{\text{Theorem}}$ Extreme Value Theorem (Weak Version)
Let $I=[a,b]$ & let $f: I \to \mathbb{R}$ be continuous on I. Then f has both an absolute minimum on I & an absolute maximum on I.

Proof: The previous theorem tells us that $f(I)$ is bounded meaning both $x = \sup(f(I))$ as well as $y = \inf(f(I))$ exist (By the completeness of $\mathbb{R}$)

[NTS: $x \in f(I)$]

Note that $x - \frac{1}{n} < x \ \forall n \in \mathbb{N}$ Hence, $\exists\, x_n \in I$ such that $x - \frac{1}{n} < f(x_n) \leq x \ \forall n \in \mathbb{N}$. Then, since I is bounded, (x_n) must also be bounded. Thus, there exists a subsequence (x_{n_k}) of (x_n) that is convergent to some $z \in \mathbb{R}$ (by the Bolzano-Weierstrass theorem). However, $z \in I$ since $x_{n_k} \in I \ \forall k \in \mathbb{N}$. Finally, we have $\lim(f(x_{n_k})) = f(z)$. But this implies that $x - \frac{1}{n} < f(x_{n_k}) \leq x$
$\Rightarrow f(z) = x$ (By the squeeze Theorem) $\Rightarrow x \in f(I)$

[NTS: $y \in f(I)$]

$y < y + \frac{1}{n} \ \forall n \in \mathbb{N} \Rightarrow \exists\, x_n \in I : y \leq f(x_n) < y + \frac{1}{n} \ \forall n \in \mathbb{N}$

Since (x_n) is bounded there exists some subsequence (x_{n_k}) of (x_n) that converges to some $z \in \mathbb{R}$ (by the Bolzano-Weierstrass theorem). However, $z \in I$ since $x_{n_k} \in I \ \forall k \in \mathbb{N}$. Therefore, we have $\lim(f(x_{n_k})) = f(z)$. But this implies that $y \leq f(x_{n_k}) < y + \frac{1}{n} \Rightarrow f(z) = y$ (by the squeeze theorem)
$\Rightarrow y \in f(I)$ ∎

Note: This is the weak version of this theorem due to the fact that there are generalzations of it that replace $[a,b]$ with any compact set (even those that are not subsets of $\mathbb{R}$).

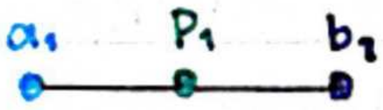
Intermediate Value Theorem

5.3 Continuous Functions on Intervals

$\boxed{\text{Theorem}}$: Intermediate Value Theorem

Let I be an interval (it need not be closed nor bounded necessarily) & let $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ & $k \in (f(a), f(b))$ then there exists $c \in (\min\{a,b\}, \max\{a,b\})$ such that $f(c) = k$.

Proof: WLOG, assume $a < b$ & consider $g: I \to \mathbb{R}$ via $g(x) = f(x) - k$. Then it is enough to show that $\exists c \in (a,b)$ such that $g(c) = 0$. Define $I_1 := [a_1, b_1]$ where $a_1 = a$, $b_1 = b$ & $p_1 = \frac{a_1 + b_1}{2}$ ← midpoint

I_1: 

If $g(p_1) = 0$, then $c = p_1$ & we are done.

Otherwise, either $\begin{cases} g(p_1) > 0 \Rightarrow \text{set } a_2 = a_1,\ b_2 = p_1 \\ \text{or} \\ g(p_1) < 0 \Rightarrow \text{set } a_2 = p_1,\ b_2 = b_1 \end{cases}$

In both cases set $p_2 = \frac{a_2 + b_2}{2}$ & $I_2 = [a_2, b_2]$; $g(a_2) < 0 < g(b_2)$.

We can continue this process to obtain intervals $I_1, I_2, \ldots, I_m$ where each $I_\ell = [a_\ell, b_\ell]$, $p_\ell = \frac{a_\ell + b_\ell}{2}$ & $g(a_\ell) < 0 < g(b_\ell)$ $\forall \ell \in \mathbb{N}_m$ for some $m \in \mathbb{N}$.

Then, if $g(p_m) = 0$, then set $c = p_m$ & we are done.

Otherwise, either $\begin{cases} g(p_m) > 0 \Rightarrow \text{set } a_{m+1} = a_m,\ b_{m+1} = p_m \\ \text{or} \\ g(p_m) < 0 \Rightarrow \text{set } a_{m+1} = p_m,\ b_{m+1} = b_m \end{cases}$

$p_{m+1} = \frac{a_{m+1} + b_{m+1}}{2}$ & $I_{m+1} = [a_{m+1}, b_{m+1}]$. If this process terminates then $\exists m \in \mathbb{N}$ such that $c = p_m$ & we are done. Otherwise we obtain a series of nested intervals $I_1 \subseteq I_2 \subseteq \cdots$ such that the length of I_n is given by $\frac{b-a}{2^n}$ along with the monotone sequences (a_n) & (b_n) satisfying $a_n \le b_n$ & $g(a_n) \le g(b_n)$ $\forall n \in \mathbb{N}$.

The nested interval property of $\mathbb{R}$ implies that $\bigcap_{n \in \mathbb{N}} I_n = \{w\}$ for some $w \in I$. Moreover,

$$0 \le w - a_n \le b_n - a_n = \frac{b-a}{2^n} \Rightarrow \lim(a_n) = w = \lim(b_n).$$

Since g is continuous at w $\begin{cases} g(w) = \lim(g(a_n)) \le 0 \\ g(w) = \lim(g(b_n)) \ge 0 \end{cases} \Rightarrow g(w) = 0$ & $w = c \Rightarrow g(c) = 0$

Hence $f(w) = k$. ∎

5.3 Continuous Functions on Intervals

There is a more general phenomenon underpinning the intermediate value theorem: *Continuous functions send connected sets to connected sets.* This fact holds true for topological spaces in general.

The proof on the previous page allows us to approximate solutions to $f(x)=k$ if the necessary assumptions are satisfied, as long as we have a method of evaluating f at each a_n & b_n. However, another subtle issue lies in checking that the error term $\underbrace{f(P_n)-k}_{g(P_n)}$ is getting $\underbrace{\text{small}}_{\substack{\text{whatever} \\ \varepsilon>0 \text{ you pick}}}$ within a practical $\underbrace{\text{number of iterations.}}_{\substack{\text{whatever } N \text{ is necessary} \\ \text{to get } |g(P_n)|<\varepsilon \ \forall n \geq N}}$

$\boxed{\text{Theorem}}$: Let I be a closed bounded interval & let $f: I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is a closed bounded interval.

Proof: $f(I) \subseteq [\inf(f(I)), \sup(f(I))]$ (by the extreme value theorem)

(by the intermediate value theorem) $\Rightarrow [\inf(f(I)), \sup(f(I))] \subseteq f(I)$ ■

Note: we will not necessarily have $f([a,b]) = \begin{cases} [f(a), f(b)] \\ [f(b), f(a)] \end{cases}$ or, unless f is monotone.

$\boxed{\text{Theorem}}$: Let I be an interval & let $f: I \to \mathbb{R}$ be continuous, then $f(I)$ is an interval.

Proof: Let $\alpha, \beta \in f(I)$. Then either $[\alpha, \beta] \subseteq f(I)$ or $[\beta, \alpha] \subseteq f(I)$. The only sets satisfying this properties are intervals by definition. ■

Uniform Continuity

5.4 Uniform continuity

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$. f is **uniformly Continuous on A** if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $x \in V_\delta(u) \cap A \Rightarrow f(x) \in V_\varepsilon(u)$ for any $u \in A$.

$\boxed{\text{Theorem}}$ Let $A \subseteq \mathbb{R}$ & let $f: A \to \mathbb{R}$. Then the following are equivalent:

1. f is <u>NOT</u> uniformly continuous on A.
2. $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists u \in A$ & some $x \in V_\delta(u) \cap A$ for which $f(x) \notin V_\varepsilon(f(u))$.
3. $\exists \varepsilon > 0$ & two sequences $(x_n), (y_n)$ in A such that $\lim(x_n - y_n) = 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ $\forall n \in \mathbb{N}$

Proof:

(1.$\Rightarrow$2.): "not uniformly continuous" means $\exists \varepsilon > 0$ such that $\forall \delta > 0$ we can find $x, u \in A$ with $|x - u| < \delta$ & $|f(x) - f(u)| \geq \varepsilon$ by definition.

(2.$\Rightarrow$3.) Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$ let $\delta := 1/n$ to obtain $u_n, x_n \in A$ such that $|x_n - u_n| < 1/n$ & $|f(x_n) - f(u_n)| \geq \varepsilon$. Set $y_n := u_n$, then $|x_n - y_n| < 1/n$ hence: $\lim(x_n - y_n) = 0$, while $|f(x_n) - f(y_n)| \geq \varepsilon$ $\forall n \in \mathbb{N}$.

(3.$\Rightarrow$1.) Assume 3. If f were uniformly continuous then $\exists \delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $x, y \in A$. Set $N \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq N$, & $|f(x_n) - f(y_n)| < \varepsilon$, but this would contradict our assumption of 3. meaning f must not be uniformly continuous. $\blacksquare$

$\boxed{\text{Theorem}}$ Let I be a closed bounded interval & let $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Proof: Suppose for a contradiction that f is not uniformly continuous on I. Then $\exists \varepsilon > 0$ as well as two sequences (x_n) & (y_n) in I such that $|x_n - y_n| < 1/n$ & $|f(x_n) - f(y_n)| \geq \varepsilon$ $\forall n \in \mathbb{N}$. Since I is bounded $\Rightarrow (x_n)$ is bounded meaning there exists a subsequence (x_{n_k}) of (x_n) that converges to some $z \in \mathbb{R}$ (by the Bolzano-Weierstrass theorem). Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$ If $\varepsilon' > 0$, then $\exists N_1 \in \mathbb{N}$ such that $1/N_1 < \varepsilon'$. Furthermore, $\exists N_2 \in \mathbb{N}$ such that $|x_{n_k} - z| < \varepsilon'/2$ $\forall k \geq N_2$. So $k \geq \max\{N_1, N_2\} := N$

$\Rightarrow |z - y_{n_k}| \leq |z - x_{n_k}| + |x_{n_k} - y_{n_k}| < \varepsilon'/2 + 1/N < \varepsilon'$

$\Rightarrow \lim(y_{n_k}) = z = \lim(x_{n_k})$

Thus, since f is continuous at z we must have $\lim(f(x_{n_k}) - f(y_{n_k})) = f(z) - f(z)$ despite $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$. But this is a contradiction since $\varepsilon > 0$. Hence, f must be uniformly continuous on I. ∎

If we want an even stricter condition than uniform continuity we can turn to what is called *Lipschitz continuity*.

Let $A \subseteq \mathbb{R}$ & $f: A \to \mathbb{R}$ if $\exists k > 0$ such that $|f(x) - f(u)| \leq k|x - u|$ $\forall x, u \in A$ then f is **Lipschitz**

$\boxed{\text{Theorem}}$: *Compositions of Continuous Functions*

Let $A, B \subseteq \mathbb{R}$ & let $f: A \to \mathbb{R}$ & $g: B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. Then if f is continuous at c & g is continuous at $f(c)$ then $g \circ f$ is continuous at c.

<u>*Proof:*</u> Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ such that

$y \in B \cap V_\delta(f(c)) \Rightarrow g(y) \in V_\varepsilon((g \circ f)(c))$

But $\delta > 0 \Rightarrow \gamma > 0$ such that

$x \in A \cap V_\gamma(c) \Rightarrow f(x) \in V_\delta(f(c))$. Hence,

$x \in A \cap V_\gamma(c) \Rightarrow (g \circ f)(x) \in V_\varepsilon((g \circ f)(c))$. ■

$\boxed{\text{Example: Lipschitz Continuity}}$ If $f: [0,1] \to \mathbb{R}$ via $x \mapsto x^2$ then $|x^2 - c^2| = |x - c|\underbrace{|x + c|}_{\leq 2 \;\because\; x, c \in [0,1]} \leq 2|x - c|$ $\Rightarrow$ f is lipschitz continuous.

But, if $f: [0,1] \to \mathbb{R}$ via $x \mapsto x^{1/2}$ then f is <u>NOT</u> Lipschitz since

$$\frac{\left|\sqrt{0} - \sqrt{\frac{1}{n^2}}\right|}{\left|0 - \frac{1}{n^2}\right|} = \frac{1/n}{1/n^2} = n$$

which implies that if $k > 0$ is fixed then, $n \geq \lceil k \rceil \Rightarrow \left|0 - \sqrt{\frac{1}{n^2}}\right| \not\leq k \left|0 - \sqrt{\frac{1}{n^2}}\right|$

$\boxed{\text{Theorem}}$: If $f: A \to \mathbb{R}$ is Lipschitz on A then f is uniformly continuous on A.

<u>*Proof:*</u> Let $\varepsilon > 0$ & $\delta = \frac{\varepsilon}{k}$ where k is a Lipschitz constant of f. Then

$$|f(x) - f(u)| \leq k|x - u| < k \underbrace{\frac{\varepsilon}{k}}_{\delta} = \varepsilon$$

whenever $x \in V_\delta(u) \cap A$. ■

$\boxed{\text{Theorem}}$: A function $f:(a,b)\to\mathbb{R}$ is uniformly continuous on (a,b) $\Leftrightarrow$ There exists a continuous function $F:[a,b]\to\mathbb{R}$ where $F|_{(a,b)}=f$.

<u>*Proof:*</u> ($\Leftarrow$) True by definition of uniform continuity.

($\Rightarrow$) Suppose that $f:(a,b)\to\mathbb{R}$ is uniformly continuous on (a,b). Now, consider any sequence (a_n) in (a,b) such that $\lim(a_n)=a$. Then (a_n) is cauchy & hence so is $(f(a_n))$. Thus, $\lim(f(a_n))=L_1$ exists. Similarly if (b_n) is any sequence in (a,b) such that $\lim(b_n)=b$ from which it follows that $\lim(f(b_n))=L_2$ also exists. So, set

$$F:[a,b]\to\mathbb{R} \text{ via } x\mapsto\begin{cases} L_1 & \text{if } x=a \\ L_2 & \text{if } x=b \\ f(x) & \text{if } a<x<b \end{cases}$$

If F is continuous on $[a,b]$ then we are done. Otherwise, let (u_n) be any sequence in (a,b) such that $\lim(u_n)=a$. Then, we have: $\lim(u_n-a_n)=a-a=0$ & so the uniform continuity of f we have that:

$$\begin{aligned}\lim(F(u_n)) &= \lim(f(u_n)) \\ &= \lim(f(u_n)-f(a_n))+\lim(f(a_n)) = 0+L_1\end{aligned}$$

Similarly if (v_n) is any sequence in (a,b) such that $\lim(v_n)=b$

$$\begin{aligned}\lim(F(v_n)) &= \lim(f(v_n)) \\ &= \lim(f(v_n)-f(b_n))+\lim(f(b_n)) = 0+L_2\end{aligned}$$

$\therefore$ F is continuous on $[a,b]$ by the sequential criterion of continuity. ■

Let I & $s: I \to \mathbb{R}$, s is a **step function** if there exists a countable partition of I into intervals such that s is constant on each section of the countable partition.

$\boxed{\text{Theorem}}$ Let I be a closed bounded interval & let $f: I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$ then there exists a step function $s_\varepsilon : I \to \mathbb{R}$ such that:

$$|f(x) - s_\varepsilon(x)| < \varepsilon \quad \forall\, x \in I \qquad [\text{where } I = [a,b]]$$

Proof: f is uniformly continuous on I which implies that $\exists\, \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$. So.

$$I = \Big(\bigcup_{k=2}^{N} \underbrace{\left(a + (k-1)\left(\frac{b-a}{N}\right),\ a + k\left(\frac{b-a}{N}\right)\right)}_{I_k \text{ for } k \in \mathbb{N}_N \setminus \{1\}}\Big) \cup \underbrace{\left[a,\ a + \frac{b-a}{N}\right]}_{I_1}$$

Where

$I_i \cap I_j \neq \emptyset \iff i = j$. Since the length of each I_k is less than δ, it follows that: $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I_k$. Hence setting $s_\varepsilon : I \to \mathbb{R}$ via $x \mapsto f\left(a + k\left(\frac{b-a}{N}\right)\right)$

Then, if $x \in I_k$ then

s_ε is a step function & $|s_\varepsilon(x) - f(x)| < \varepsilon, \forall x \in I$ ∎

Monotone & Inverse Functions

5.6: Monotone & Inverse Functions

Let $f: A \to \mathbb{R}$.

f is increasing on A if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ $\forall x_1, x_2 \in A$

& strictly increasing on A if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ $\forall x_1, x_2 \in A$.

f is decreasing on A if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ $\forall x_1, x_2 \in A$

& strictly decreasing on A if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ $\forall x_1, x_2 \in A$.

↳ We say that f is **monotone** on A if f is either increasing or decreasing & **strictly monotone** on A if f is either strictly increasing or strictly decreasing.

> Note: If $g: A \to \mathbb{R}$ is decreasing on A then $f = -g$ is increasing on A. So, we generally assume our functions are increasing.

$\boxed{\text{Theorem}}$: Let I be an interval & let $f: I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I then:

① $\lim\limits_{x \to c} \left(f\big|_{I \cap (-\infty, c)}(x) \right) = \sup\left(f(I \cap (-\infty, c)) \right)$

② $\lim\limits_{x \to c} \left(f\big|_{I \cap (c, \infty)}(x) \right) = \inf\left(f(I \cap (c, \infty)) \right)$

<u>*Proof:*</u> Let $x \in I$ & $x < c$. Then $f(x) \leq f(c)$. Hence $s = \sup(\{f(x) : x \in I, x < c\})$ exists. Let $\varepsilon > 0$. Then, $s - \varepsilon < s \Rightarrow s - \varepsilon$ is not an upper bound of $f(I \cap (-\infty, c)) \Rightarrow \exists y \in f(I \cap (-\infty, c))$ such that $s - \varepsilon < y \Rightarrow \exists x \in I \cap (-\infty, c)$ such that $y = f(x) \Rightarrow s - \varepsilon < f(x) \leq s$. Since f is increasing, it follows that if $\delta = |c - x| = c - x$ then, $z \in (I \cap (-\infty, c)) \cap V_\delta(c)$

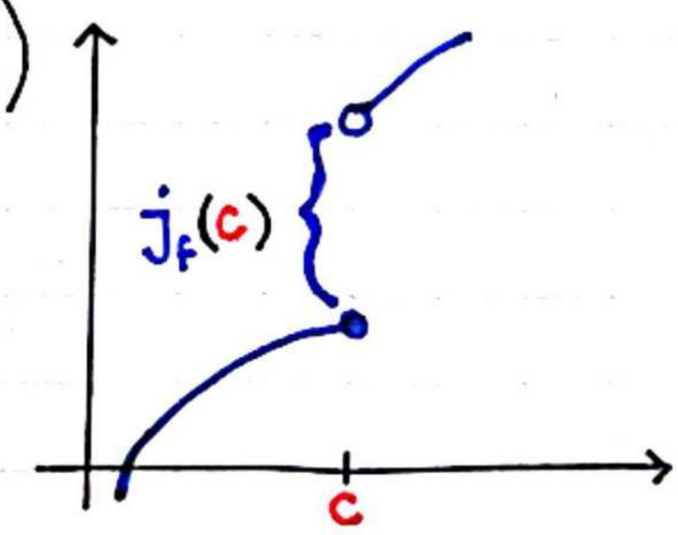
$\Rightarrow s - \varepsilon < f(x) \leq f(z) \leq s \Rightarrow f(z) \in V_\varepsilon(s)$

$\therefore \lim\limits_{x \to c} \left(f\big|_{I \cap (-\infty, c)}(x) \right) = s$.

A similar argument shows the second part. ∎

If $f: I \to \mathbb{R}$ is increasing & c is <u>NOT</u> an endpoint of I then we define $j_f(c)$ to be the **jump** of f at c where

$$j_f(c) := \lim_{x \to c}\left(f\big|_{(c,\infty)\cap I}(x)\right) - \lim_{x \to c}\left(f\big|_{(-\infty,c)\cap I}(x)\right)$$



$\boxed{\text{Theorem}}$ Let I be an interval, then

$f: I \to \mathbb{R}$ is continuous at $c \iff j_f(c) = 0$

$\boxed{\text{Theorem}}$ Let $f: I \to \mathbb{R}$ be monotone, then, the set of points where f is discontinuous is countable.

<u>*Proof:*</u> Without loss of generality, assume that f is increasing. The theom above this one tells us that:

$$D \doteq \{x \in I : j_f(x) \neq 0\} = j_f^{-1}(\mathbb{R} \setminus \{0\})$$

suppose that $c \in D$ this implies that:

$$\exists\, x_c \in \mathbb{Q} \cap \Big(\underbrace{\lim_{x\to c}\left(f\big|_{(-\infty,c)}(x)\right)}_{f(c^-)},\ \underbrace{\lim_{x\to c}\left(f\big|_{(c,\infty)}(x)\right)}_{f(c^+)}\Big)$$

↳ Since $\mathbb{Q}$ is dense in $\mathbb{R}$.

adjust accordingly if c is an endpoint of I

So, let $g: D \to \mathbb{Q}$ via $c \mapsto x_c$. Then if $c_1, c_2 \in D$ & $c_1 < c_2$ then $f(c_1^-) < \underbrace{x_{c_1}}_{} < f(c_1^+)$

$$g(c_1) \leq f(c_2^-) < x_{c_2} = g(c_2) < f(c_2^+)$$

$$\Rightarrow g(c_1) < g(c_2)$$

$\therefore$ g is injective & so D must be countable ∎

$\boxed{\text{Theorem}}$ Let I be an interval & let $f: I \to \mathbb{R}$ be strictly monotone & continuous on I with image J. Then, $f^{-1}: J \to I$ is strictly monotone & continuous on J.

Proof: Without loss of generality, assume that f is strictly increasing. Then J must be an interval. Furthermore $f^{-1}: J \to I$ is well-defined since $f: I \to J$ is a bijection ($\because$ strictly increasing $\Rightarrow$ injective)

Claim: $f^{-1}: J \to I$ is strictly increasing.

Let $y_1, y_2 \in J$ such that $y_1 < y_2$. Then $f(x_1) = y_1$ & $f(x_2) = y_2$ for some $x_1, x_2 \in I$. Since f is strictly increasing, it must be the case that $x_1 < x_2$.

$\therefore y_1 < y_2 \Rightarrow \underbrace{x_1}_{f^{-1}(y_1)} < \underbrace{x_2}_{f^{-1}(y_2)} \Rightarrow f^{-1}: J \to I$ is strictly increasing on J

Finally, we show that f^{-1} is continuous on J

Note that: f^{-1} is continuous $\iff j_{f^{-1}}(c) = \emptyset \ \forall c \in J$.

But if $\exists c \in j_{f^{-1}}^{-1}(\mathbb{R} \setminus \{\emptyset\})$ then $(f^{-1}(c^-), f^{-1}(c^+)) \cap I = \emptyset$ despite $(f^{-1}(c^-), f^{-1}(c^+)) \subseteq I$: contradiction!

$\therefore f^{-1}$ must be continuous on J. ∎

This theorem yields the continuity of many familiar inverse functions.

$\boxed{\text{example}}$ Let n be even, then $f: [0, \infty] \to [0, \infty]$ via $x \mapsto x^n$ is strictly increasing & surjective. So, by the previous theorem $f^{-1}: [0, \infty) \to [0, \infty)$ also exists & it is strictly increasing, surjective, & continuous. We define $f^{-1}(x) := \sqrt[n]{x}$ & write $x^{\frac{1}{n}} := \sqrt[n]{x}$ as well. Since $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$ we have $(x^{\frac{1}{n}})^n = x = (x^n)^{\frac{1}{n}} \ \forall x \in [0, \infty)$

Introduction to Metric Spaces

11.4: Metric Spaces

A **metric** on a set S is a function $d: S \times S \to \mathbb{R}$ that satisfies the following properties:

1. $d(x,y) \geq 0 \quad \forall x,y \in S$ (non-negativity)
2. $d(x,y) = 0 \iff x = y$ (zero at identical points)
3. $d(x,y) = d(y,x) \quad \forall x,y \in S$ (commutativity)
4. $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in S$ (triangle inequality)

A **metric space** is an orded pair (S,d) where S is a set & d is a metric on S.

$\boxed{\text{Examples}}$:

1. (A,d) where $\emptyset \subsetneq A \subseteq \mathbb{R}$ & $d(x,y) := |x-y| \quad \forall x,y \in A$ is a metric space (its what we have been working in all semester).

2. (A,d) where $\emptyset \subsetneq A \subseteq \mathbb{R}$ & $d(x,y) := \sum_{i=1}^{n} |x_i - y_i| \quad \forall x,y \in A$ is a metric space as well.

3. Let $C^0(X,\mathbb{R})$ be the set of all continuous functions $f: X \to \mathbb{R}$ that are continuous on X. Then $(C^0([0,1],\mathbb{R}), d)$ where $d(f,g) := \sup(\{|f(x)-g(x)| : x \in [0,1]\})$ is a metric space.

4. (A,d) where $\emptyset \subsetneq A \subseteq \mathbb{R}$ & $d(x,y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ with $\left.\begin{array}{l} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{array}\right\} \in A$ is a metric space.

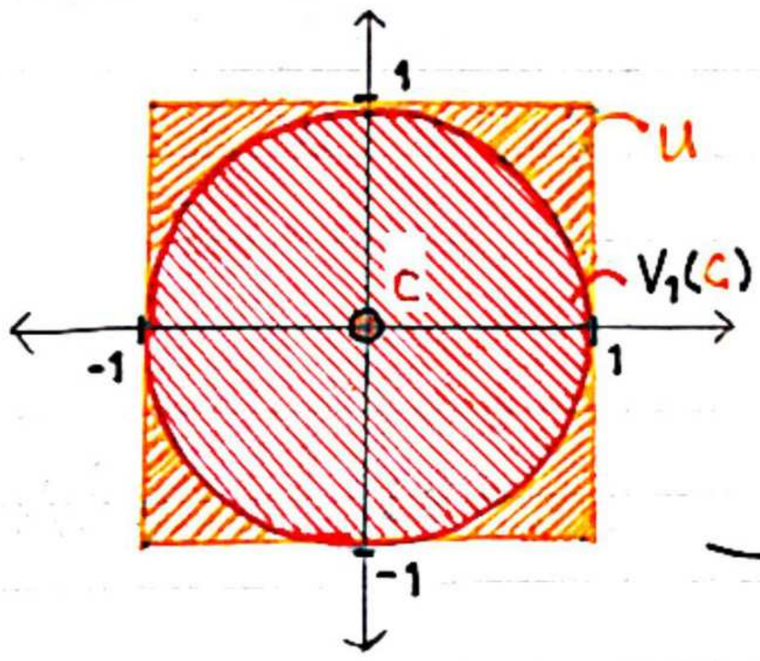
5. Let S be any non-empty set. Then (S,d) is where $d(x,y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ is a metric space. In fact this (S,d) is called a *discrete metric space*.

Neighborhoods / Balls

Let (S, d) be a metric space & $\varepsilon > 0$. Then, given any $c \in S$ we set $V_\varepsilon(c) := \{x \in S : d(x,c) < \varepsilon\}$. This is the ε-neighborhood of c or an *ε-ball around c*.

$\boxed{\text{Example}}$:

$(\mathbb{R}^2, d)$ where $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$. If $c = (0,0)$ $u = (-1,1)\times(-1,1) \Rightarrow V_1(c) \subseteq u$ meaning u is a neighborhood of c.



A **neighborhood** of c is anus set $u \subseteq S$ such that $V_\varepsilon(c) \subseteq u$ for some $\varepsilon > 0$.

Since $V_1(c) \subseteq u$, u is a neighborhood of c

Let (x_n) be a sequence in a metric space (S, d). We sau that (x_n) <u>converges</u> to $x \in S$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\underbrace{x_n \in V_\varepsilon(x)}_{d(x_n, x) < \varepsilon}$ whenever $n \geq N$

We sau that (x_n) is <u>cauchu</u> if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n \geq m \geq N$.

Combining these two properties, a metric space (S, d) is **complete** if everu cauchu sequence in (S, d) converges to a point in S.

Open & Closed Sets

Let (S, d) be a metric space. If $G \subseteq S$, then we say that G is **open** in S if $\forall x \in S$, there exists some neighborhood u of x such that $u \subseteq G$; If $S \setminus G$ is open, then G is **closed** in S.

Example

Consider $(\mathbb{R}, d)$ where $d(x,y) := |x-y|$ $\forall x, y \in \mathbb{R}$.
$[0,1)$ is <u>NOT</u> open in $\mathbb{R}$ since $V_\varepsilon(0) \not\subseteq [0,1)$, $\forall \varepsilon > 0$.
However, $[0,1)$ is open in $[0,1]$ if we restrict d to $[0,1] \times [0,1]$ since this restriction means $V_1(0) = [0,1)$.
↳ This shows that the openness/closedness of a subset is relative to the metric space that it is defined in!

Finite intersections of open sets & arbitrary unions of open sets are open.

Proof: Let $u_1, u_2, \dots, u_n$ be open sets in a metric space (S, d). Let $x \in \bigcap_{i=1}^{n} u_i$ (meaning x is a point in u_i for $i = 1, 2, \dots, n$)
Since each u_i is open, for each u_i, there exists some $\varepsilon_i > 0$ such that $V_{\varepsilon_i}(x) \subseteq u_i$. Since $i = 1, 2, \dots, n$ is finite we can let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ which means $V_\varepsilon(x) \subseteq u_i$ for $i = 1, 2, \dots, n$ $\Rightarrow V_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} u_i$ $\forall x \in \bigcap_{i=1}^{n} u_i$
Thus, the intersection of $u_1, u_2, \dots, u_n$ is open

Now, let $x \in \bigcup_{i=1}^{\infty} u_i$ meaning there exists $i_0 \in \{1, 2, \dots\}$ such that $x \in u_{i_0}$. Since u_{i_0} is open, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq u_{i_0}$. Thus, $V_\varepsilon(x) \subseteq u_{i_0} \subseteq \bigcup_{i=1}^{\infty} u_i$ $\forall x \in \bigcup_{i=1}^{n} u_i$. Hence the union of $u_1, u_2, \dots$ is open. ∎

Let (S_1, d_1) & (S_2, d_2) be metric spaces & let $f: S_1 \to S_2$ be a function. Then f is *continuous* at $c \in S_1$ if & only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(x))$.

$\boxed{\text{Theorem}}$: $f: S_1 \to S_2$ is continuous if & only if $f^{-1}(U)$ is open whenever U is open in S_2.

A metric space (S, d) is **compact** if whenever $S = \bigcup_{\lambda \in \Lambda} U_\lambda$ where Λ is any indexing set & each U_λ is open in S, we have that $S = \bigcup_{i=1}^{n} U_{\lambda_i}$ for some finite subset $\{\lambda_i : i \in \mathbb{N}_n\} \subseteq \Lambda$.

$\boxed{\text{Theorem}}$ If (S, d) is a compact metric space & $f: (S, d) \to (S', d')$ is continuous, then, $f(S)$ is compact in S'.

A **semi-metric** on a set S is a function $f: S \times S \to \mathbb{R}$ that satisfies all of the *metric conditions* <u>except</u> that "$d(x,y) = 0 \Leftrightarrow x = y$" is replaced with "$d(x,x) = 0$ only".

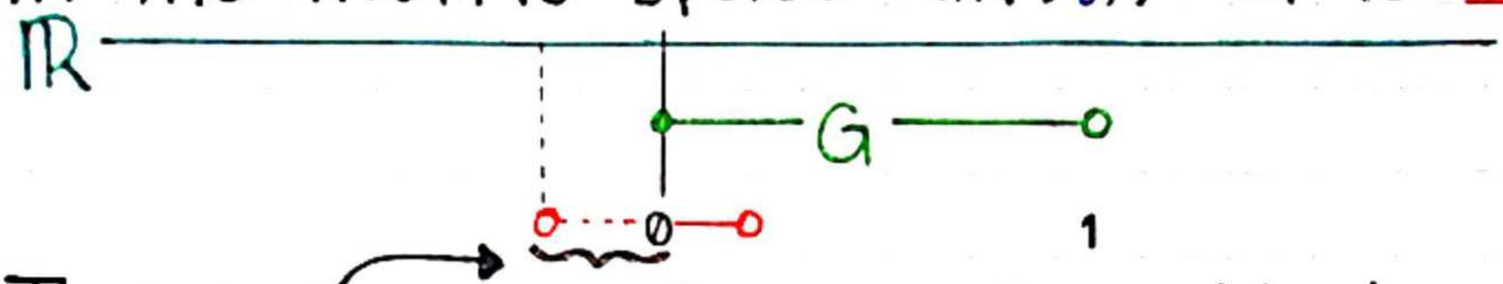
↳ Integral operators tend to yield semi-metrics.

Let (S,d) be a *metric space* & let $G \subseteq S$. We say that G is **open** in S if for every point $x \in G$, there exists some neighborhood U of x such that $U \subseteq G$.

↳ Whether or not a set is open depends on the metric space in which it is situated.

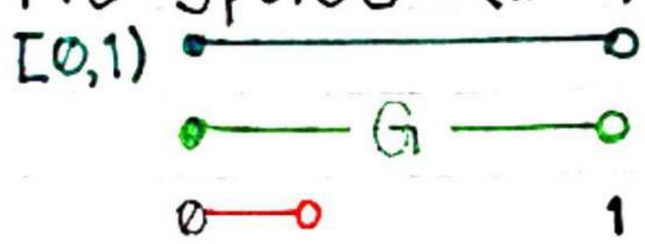
Consider $G = [0,1)$ & $d(x,y) = |x-y|$

In the metric space $(\mathbb{R}, d)$ G is <u>NOT</u> open.



This gap means for any ε-neighborhood would leak out of G into $\mathbb{R}$ meaning G is not open in $\mathbb{R}$

In the metric space $([0,1), d)$, however, G is open!



Since any ε-neighborhood will not contain any points in $[0,1)$ that are not in G, G is open in $[0,1)$

A set is open if the neighborhoods are free to traverse any available point in the metric space; ie there aren't any values in the metric space that are "closed off from" the neighborhoods by the bounds of G

Derivatives

6.1: The Derivative

Let $I \subseteq \mathbb{R}$ be an interval & let $f: I \to \mathbb{R}$ & $c \in I$, then $L \in \mathbb{R}$ is the **derivative** of f at c if given any $\varepsilon > 0$, $\exists \delta > 0$ such that $\left| \frac{f(x)-f(c)}{x-c} - L \right| < \varepsilon$ whenever $x \in (V_\delta(c) \setminus c) \cap I$

If such an L exists then we say that f is ***differentiable*** at c & we can write $f'(c) = L$; that is,

$$f'(c) = \lim_{x \to c} \left(\frac{f(x)-f(c)}{x-c} \right) \text{ whenever this limit exists}$$

> <u>Note:</u> In more general settings (like $f: \mathbb{R}^n \to \mathbb{R}^m$ or $f: A \to B$ where A & B are Banach spaces), the derivative of f at c is defined as the closest linear transformation to f at c. In our setting this corresponds to a 1×1 matrix (obv. after the basis has been chosen): $[f'(c)]$.

$\boxed{\text{Theorem 6.1}}$: If $f: I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

<u>*Proof*</u>: Since $f'(c) = \lim_{x \to c} \left(\frac{f(x)-f(c)}{x-c} \right)$ it follows that,

$$\lim_{x \to c} (f(x) - f(c))$$

$$= \lim_{x \to c} \left(\left(\frac{f(x)-f(c)}{x-c} \right)(x-c) \right) = f'(c) \cdot \underbrace{\lim_{x \to c} (x-c)}_{=0}$$

$$= 0$$

$$\therefore \lim_{x \to c} (f(x)) = f(c)$$

$\blacksquare$

Continuity is <u>NOT</u> enough to ensure differentiability. In fact, most continuous functions are not differentiable at any point.

$\boxed{\text{Theorem}}$ Let $I \subseteq \mathbb{R}$ be an interval, $\alpha \in \mathbb{R}$, & $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. Then:

- αf, $f+g$, & fg are differentiable at c
- $(\alpha f)'(c) = \alpha f'(c)$, & $(f+g)'(c) = f'(c) + g'(c)$
- $(fg)'(c) = (f'g + fg')(c) = f(c)g'(c) + f'(c)g(c)$

Furthermore, if $g(c) \neq 0$ then f/g is differentiable at c & $\left(\frac{f}{g}\right)'(c) = \left(\frac{f'g - fg'}{g^2}\right)(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$

Proof: **Scalar multiple:**

$$(\alpha f)'(c) = \lim_{x \to c}\left(\frac{\alpha f(x) - \alpha f(c)}{x - c}\right) = \alpha \lim_{x \to c}\left(\frac{f(x) - f(c)}{x - c}\right) = \alpha f'(c)$$

Sum:

$$(f+g)'(c) = \lim_{x \to c}\left(\frac{(f+g)(x) - (f+g)(c)}{x - c}\right)$$

$$= \lim_{x \to c}\left(\frac{f(x) - f(c)}{x - c}\right) + \lim_{x \to c}\left(\frac{g(x) - g(c)}{x - c}\right)$$

$$= f'(c) + g'(c)$$

Product:

$$(fg)' = \lim_{x \to c}\left(\frac{f(x)g(x) - f(c)g(c)}{x - c}\right)$$ add & subtract $f(x)g(c)$ to get

$$= \lim_{x \to c}\left(\frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c}\right)$$

$$= \lim_{x \to c}\left(\frac{f(x)g(x) - f(x)g(c)}{x - c}\right) + \lim_{x \to c}\left(\frac{f(x)g(c) - f(c)g(c)}{x - c}\right)$$

$$= \lim_{x \to c}\left(f(x)\frac{g(x) - g(c)}{x - c}\right) + \lim_{x \to c}\left(g(c)\frac{f(x) - f(c)}{x - c}\right)$$

$$= f(c)g'(c) + g(c)f'(c)$$

(continued on next page...)

Proof: (cont.) **Reciprocal:** Assume $g(c) \neq 0$. By the continuity of g at c, there exists a neighborhood of c where $g(x) \neq 0$. Let $x \neq c$ be in this neighborhood:

$$\lim_{x \to c} \left(\frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x - c} \right) = \lim_{x \to c} \left(\frac{g(c) - g(x)}{g(x)\, g(c)\, (x - c)} \right)$$

$$= \lim_{x \to c} \left(\frac{-1}{g(x)\, g(c)} \, \frac{g(x) - g(c)}{x - c} \right) \text{ Thus } \left(\frac{1}{g}\right)'(c) = \frac{-g'(c)}{(g(c))^2}$$

Quotient:

$$\left(\frac{f}{g}\right)'(c) = \left(f \cdot \frac{1}{g}\right)'(c) = f'(c) \frac{1}{g(c)} + f(c) \left(\frac{1}{g}\right)'(c)$$

$$= \frac{f'(c)}{g(c)} - f(c) \frac{g'(c)}{(g(c))^2} = \frac{f'(c)\, g(c) - f(c)\, g'(c)}{(g(c))^2} \qquad \blacksquare$$

The differentiability of $f+g$ & αf tells us that the set of functions that are differentiable on I, say $\mathcal{D}(I)$, is a real, infinite dimensional vector space. Furthermore, the derivative operator is a linear transformation from $\mathcal{D}(I)$ to $\mathcal{F}(I)$, the set of functions with domain I.

The product rule tells us that $\mathcal{D}(I)$ is actually an $\mathbb{R}$-algebra, more specifically a ring, & that the derivative operator is a linear derivation.

Proving the chain rule is way more involved than the previous theorem. We will embark on the next page in order to demonstrate some important properties of the derivatives of inverse functions. This journey will also ultimately lead us to implicit differentiation via the inverse function theorem.

$\boxed{\text{Theorem}}$: **Caratheodory's Theorem**

Let f be defined on an interval I containing c. Then f is differentiable at c if & only if $\exists \Phi: I \to \mathbb{R}$ that is continuous at c & satisfies the following: $f(x) - f(c) = \Phi(x)(x-c)$. In particular $\Phi(c) = f'(c)$.

Proof: ($\Rightarrow$): If f is differentiable at c, then set

$$\Phi(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{if } x \neq c,\ x \in I \\ f'(c) & \text{if } x = c \end{cases}$$

Then $\Phi: I \to \mathbb{R}$ is continuous on $I \setminus \{c\}$ by the algebraic continuity theorems & Φ is continuous at c since $\lim_{x \to c}(\Phi(x)) = f'(c) = \Phi(c)$ so Φ is continuous on I. Moreover, $(x-c)\Phi(x) = f(x) - f(c)$ $\forall x \in I$

($\Leftarrow$): Assume such a $\Phi: I \to \mathbb{R}$ exists. Then, $\lim_{x \to c}(\Phi(x)) = \Phi(c) \Rightarrow \lim_{x \to c}\left(\dfrac{f(x)-f(c)}{x-c}\right) = \Phi(c)$

$\therefore f'(c)$ exists & must be equal to $\Phi(c)$. ∎

$\boxed{\text{example}}$: Let $f: \mathbb{R} \to \mathbb{R}$ via $x \mapsto x^3$. Then:

$$x^3 - c^3 = \underbrace{(x^2 + cx + c^2)}_{\Phi(x)}(x-c) \Rightarrow f \text{ is differentiable at } c \text{ \& } f'(c) = \Phi(c) = 3c^2$$

$\boxed{\text{Theorem}}$: **Chain Rule** – Let $I, J \subseteq \mathbb{R}$ be intervals & let $f: J \to \mathbb{R}$, $g: I \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, f is differentiable at $c \in J$, & g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c & $(g \circ f)'(c) = g'(f(c))f'(c)$.

Proof: Caratheodory's theorem $\Rightarrow \exists \Phi: J \to \mathbb{R}$ such that Φ is continuous at c & $f(x) - f(c) = \Phi(x)(x-c)$ $\forall x \in J$.
Applying Caratheodory's theorem *again*...
$\Rightarrow \exists \Psi: I \to \mathbb{R}$ such that Ψ is continuous at $f(c)$ & $g(y) - g(f(c)) = \Psi(y)(y - f(c))$ $\forall y \in I$. In particular, this equation holds $\forall y \in f(J) \subseteq I$ meaning

$$\begin{aligned} g(f(x)) - g(f(c)) &= \Psi(f(x))(f(x) - f(c)) \\ &= (\Psi \circ f)(x)(f(x) - f(c)) \\ &= (\Psi \circ f)(x)\Phi(x)(x-c) \quad \forall x \in J \end{aligned}$$

Since $(\Phi \circ f) \cdot \Phi$ is continuous at c, Caratheodory's theorem $\Rightarrow g \circ f$ is differentiable at c & $(g \circ f)'(c) = (\Psi \circ f)(c)\Phi(c) = \Psi(f(c))f'(c) = g'(f(c))f'(c)$. ∎

Suppose that f & f^{-1} are differentiable wherever necessary, then $(f \circ f^{-1})(x) = x \Rightarrow f'(f^{-1}(x))(f^{-1})'(x) = 1$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

↳ ex: $f: x \mapsto x^2 \Rightarrow (f^{-1})'(x) = \dfrac{1}{2f^{-1}(x)} = \dfrac{1}{2\sqrt{x}}$

The theorem on the next page formallizes this example in a way that we can properly generallize.

6.1 Intro to derivatives

$\boxed{\text{Theorem}}$: Let I be an interval & let $f: I \to \mathbb{R}$ be strictly monotone & continuous on I. Let $J = f(I)$, then, $f^{-1}: J \to I$ is differentiable at $f(c)$ if $c \in I$ <u>and</u> f is differentiable at c <u>and</u> $f'(c) \neq 0$. In this case,

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

<u>*Proof:*</u> Let $c \in I$. Then, by Caratheodory's theorem $\exists \varphi: I \to \mathbb{R}$ such that φ is continuous at c & $f(x) - f(c) = \varphi(x)(x-c)$ $\forall x \in I$ & $\varphi(c) = f'(c)$. Since $f'(c) \neq 0$, we have $\varphi(c) \neq 0$. So, $\exists \delta > 0$ such that $x \in I \cap V_\delta(c) \Rightarrow \varphi(x) \neq 0$. Let $u = f(I \cap V_\delta(c))$. Since $I \cap V_\delta(c)$ is an interval we have $f(f^{-1}(y)) = y$ $\forall y \in u$. So, $y - f(c) = f(f^{-1}(y)) - f(c)$.
$= \varphi(f^{-1}(y))(f^{-1}(y) - f^{-1}(f(c)))$

6.2 The Mean Value Theorem

Let I be an interval. Then $f: I \to \mathbb{R}$ has a **relative maximum** at $c \in I$ if $\exists \delta > 0$ such that $x \in V_\delta(c) \cap I \Rightarrow f(x) \leq f(c)$ & f has a **relative minimum** at $c \in I$ if $\exists \delta > 0$ such that $x \in V_\delta(c) \cap I \Rightarrow f(x) \geq f(c)$. In either case, we say f has a **relative extremeum** at c.

$\boxed{\text{Theorem}}$: **Fermat's Theorem**

Let c be an interior point of an interval I & suppose $f: I \to \mathbb{R}$ has a relative extremum at c. If f is differentiable at c then $f'(c) = 0$.

Proof: Assume (WLOG) that f has a relative max at c. If $f'(c) > 0$, then $\exists \delta > 0$ such that $x \in V_\delta(c) \cap I \setminus \{c\} \Rightarrow \frac{f(x)-f(c)}{x-c} > 0$. But!

If $x \in (V_\delta(c) \setminus \{c\}) \cap I$ & $x > c$, then $x - c > 0$ & $f(x) - f(c) \leq 0 \Rightarrow \frac{f(x)-f(c)}{x-c} \leq 0$ contradiction!

If $f'(c) < 0$, then $\exists \delta > 0$ such that $x \in (V_\delta(c) \setminus \{c\}) \cap I \Rightarrow \frac{f(x)-f(c)}{x-c} < 0$ But!

If $x \in (V_\delta(c) \setminus \{c\}) \cap I$ & $x < c$, then $x - c > 0$ & $f(x) - f(c) \geq 0 \Rightarrow \frac{f(x)-f(c)}{x-c} > 0$ contradiction!

$\therefore f'(c) = 0$ ∎

It follows from this theorem that if $f: I \to \mathbb{R}$ has a relative extremum at an interior point $c \in I$ then $f'(c) = 0$ or f is non-differentiable at c.

6.2 The Mean Value Theorem

Theorem **Rolle's Theorem**

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ & differentiable on (a,b). If $f(a)=f(b)=0$ then, $\exists c\in(a,b)$ such that $f'(c)=0$.

Proof: If $f:x\mapsto 0$ $\forall x\in[a,b]$ then we are done. Otherwise, WLOG assume that f has a positive output. This implies that $M=\max(\{f(x):x\in[a,b]\})$ exists by the extreme value theorem & M is positive. Let $c\in[a,b]$ such that $f(c)=M$. Since $M>0$, it follows from $f(a)=f(b)=0$ that $c\neq a$ & $c\neq b$. Hence $c\in(a,b)$ meaning $f'(c)$ exists. Thus, the previous theorem implies that $f'(c)=0$. ∎

Theorem **The Mean Value Theorem**

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ & differentiable on (a,b). Then $\exists c\in(a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a)$$

Proof: Apply Rolle's theorem to the function $\Phi:[a,b]\to\mathbb{R}$ defined by

$$\Phi(x)=f(x)-f(a)-\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\quad\forall x\in[a,b]$$ ∎

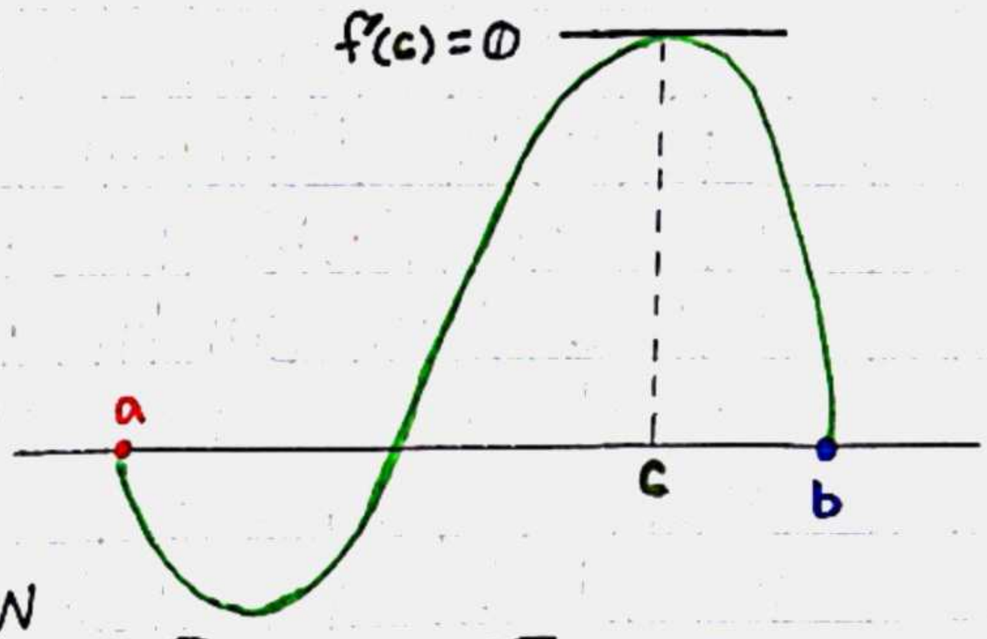
The mean value theorem tells us that the average rate of change over an interval had to occurr at least once as an instantaneous rate of change

6.2 The mean value Theorem

Rolle's theorem & the mean value theorem tell us about the intermediate behavior of a smooth function by fixing the end points of an interval. Essentially, by "teathering" a smooth function at two points the function loses its freedom to do whatever it wants in between them.

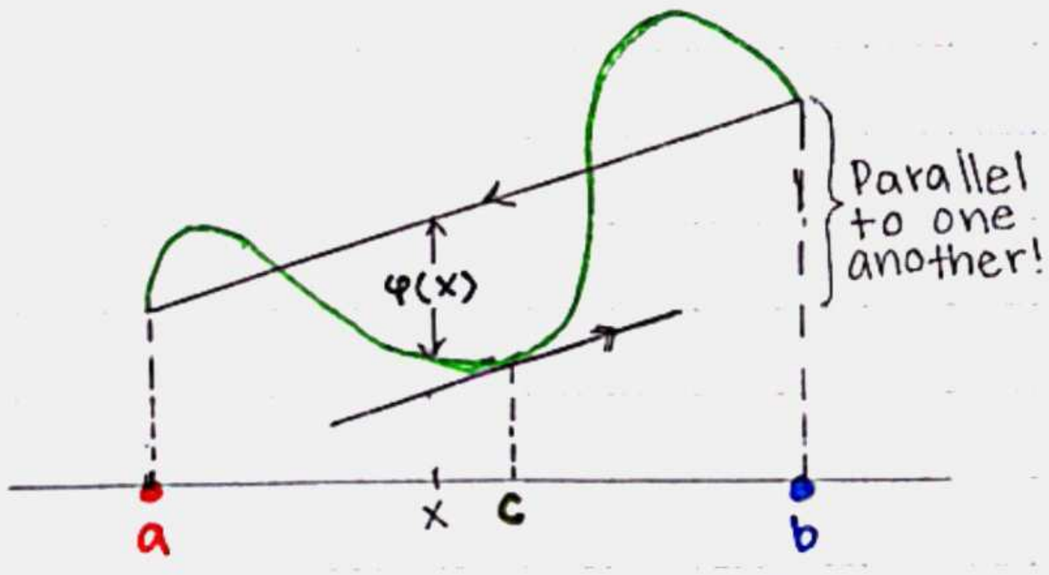
Rolle's Theorem - "what goes up must come down"

Tells us that if a function f is smooth & equal to zero at both a & b, then there *must* be at least one critical point between a & b. To throw a ball upwards into the air (from the ground) & have it fall back to the ground, there necessarily must be a point at which the ball stops moving upwards & starts falling back down. If we treat the path of this ball as a function of time then this point where the ball starts falling is the point who's derivative is equal to zero that Rolle's theorem tells us must exist.



Rolle's Theorem

The mean value theorem is *literally* just a tilted version of Rolle's theorem; to prove the mvt we just reverse this tilt & then apply Rolle's theorem.



The Mean Value Theorem

6.2 The Mean Value Theorem

Rolle's Theorem: Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ & differentiable on (a,b). If $f(a)=0=f(b)$ then $\exists c\in(a,b)$ such that $f'(c)=0$

A more insightful Proof:

Step 1: The trivial case: If $f(x)=0\ \forall x\in[a,b]$, then f is a horizontal line which makes all points in the interval both relative mins & maxes thus the derivative is zero for the entire interval.

Step 2: Apply the Extreme Value Theorem: Since the interval $[a,b]$ is a (compact) closed & bounded set & f is continuous the function is *forced* to attain a maximum value; if the interval were open or f were discontinuous then it would be possible for f to diverge infinitely to an asymptote. Meaning if f is non-zero at at least one point in (a,b) then we can write

$$M=\max(\{|f(x)|: x\in[a,b]\})\quad \&$$

M exists by the extreme value theorem Furthermore $f(a)\neq M\neq f(b)$ since $M>0$.

Step 3: Find the Peak: Thus, since $f(c)=M$ is true for some $c\in(a,b)$ we know that $f(c)$ is a peak of f. Hence, Fermat's theorem tells us that the derivative of f at a relative extremum of f must be equal to zero. This means $f'(c)=0$ as desired. ∎

6.2 The Mean Value Theorem

$\boxed{\text{Theorem}}$: Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ & differentiable on (a,b). If $f'(x)=0$ $\forall x\in(a,b)$, then f is a constant function.

Proof: Let $x\in[a,b]$, then, by the mean value theorem: for $f|_{[a,x]}$ we have some $c\in(a,x)$ such that $f(x)-f(a)=f'(c)(x-a)=0$ since $f'(c)=0$ $\Rightarrow f(x)=f(a)$. ∎

$\boxed{\text{Corallary}}$: If f & g are continuous on $[a,b]$ & differentiatiable on (a,b) & $f'(x)=g'(x)$ $\forall x\in(a,b)$, then $\exists c\in\mathbb{R}$ such that $f(x)=g(x)+c$ $\forall x\in[a,b]$.

$\boxed{\text{Theorem}}$: Let $f:I\to\mathbb{R}$ be differentiable on I. Then

1. f is increasing on $I \Leftrightarrow f'(x)\geq 0$ $\forall x\in I$.
2. f is decreasing on $I \Leftrightarrow f'(x)\leq 0$ $\forall x\in I$.

Proof: We will prove 1. but a similar argument applies to 2. as well.

($\Leftarrow$) Suppose that $f'(x)\geq 0$ $\forall x\in I$. If $x_1, x_2\in I$ & $x_1<x_2$. Then, by the MVT on $[x_1,x_2]$ $\Rightarrow \exists c\in(x_1,x_2)$ such that $f(x_2)-f(x_1)=f'(c)(x_2-x_1)\geq 0$ $\therefore f(x_2)\geq f(x_1)$

($\Rightarrow$) Suppose that f is increasing on I. Then, $x,c\in I$ & $x\neq c \Rightarrow \dfrac{f(x)-f(c)}{x-c}\geq 0$.

Since $f(x)-f(c)$ & $x-c$ are either both non-negative or both non-positive. By the previous corallary we have $f'(c)=\lim\limits_{x\to c}\left(\dfrac{f(x)-f(c)}{x-c}\right)\geq 0$ ∎

6.2 The Mean Value Theorem

$\boxed{\text{Theorem}}$: **Theorem First-Derivative Test**

Let $f:[a,b]\to\mathbb{R}$ be continuous & let $c\in(a,b)$. Suppose further that f is differentiable on $(a,c)\cup(c,b)$. Then

1. If $\exists\,\delta>0$ such that $\begin{cases} x\in(c-\delta,c) \Rightarrow f'(x)\geq 0 \\ x\in(c,c+\delta) \Rightarrow f'(x)\leq 0 \end{cases}$ Then f has a relative max. at c.
2. If $\exists\,\delta>0$ such that $\begin{cases} x\in(c-\delta,c) \Rightarrow f'(x)\leq 0 \\ x\in(c,c+\delta) \Rightarrow f'(x)\geq 0 \end{cases}$ Then f has a relative min. at c.

Proof: (we will prove 1. but a similar argument applies to 2. as well)

1. Let $\delta>0$ satisfy the conditions of 1. Then, the mean value theorem implies that if $x\in(c-\delta,c)$ then $\exists\, c_x\in(x,c)$ such that $f(c)-f(x)=f'(c_x)(c-x)$ & since $f'(c_x)\geq 0$ we have $f(c)-f(x)=f'(c_x)(c-x)\geq 0$. Similarly, if $x\in(c,c+\delta)$ the mean value theorem implies that $\exists\, c_x'\in(c,x)$ such that

$$f(c)-f(x)=\underbrace{f'(c_x')}_{\leq 0}\underbrace{(c-x)}_{\leq 0}\geq 0 \Rightarrow f(c)-f(x)\geq 0 \text{ in both cases meaning...}$$

$\therefore$ f has a relative max at c.

↳ Note:

The converse of the first derivative test is not always true; ie just because $f:[a,b]\to\mathbb{R}$ has a relative extremum at c does not gurrantee that that a $\delta>0$ meeting the corresponding criteria exists for f!

6.2 The Mean Value Theorem

$\boxed{\text{Example:}}$ Let $h(x) = (1+x)^{\alpha}$ where $x \in (-1, \infty)$ & $\alpha \in (1, \infty)$

The mean value theorem tells us that if we can find $a \leq b$ such that h is continuous on $[a, b]$ & differentiable on (a,b) then $\exists c \in (a,b)$ such that

$$h(b) - h(a) = h'(c)(b-a)$$

Applying the MVT to the interval $[0, x]$ we get

$$h(x) - h(0) = h'(c)(x-0)$$

$$h(0) = (1+0)^{\alpha} = 1^{\alpha} = 1 \quad \& \quad h'(c) = \alpha(1+c)^{\alpha-1}$$

$$\Rightarrow (1+x)^{\alpha} - 1 = \alpha(1+c)^{\alpha-1}(x)$$

$$(1+x)^{\alpha} = \alpha(1+c)^{\alpha-1}(x) + 1 \quad \& \text{ since } (1+c)^{\alpha-1} > 1$$

$$\Rightarrow (1+x)^{\alpha} > \alpha x + 1$$

The same holds for $(-1, 0)$ giving us a second proof of Bernoulli's inequality.

$\boxed{\text{Lemma}}$: Let I be an interval & let $f: I \to \mathbb{R}$ be differentiable at some $c \in I$. Then,

1. If $f'(c) > 0$, then $\exists \delta > 0$ such that $f(x) > f(c)$ $\forall x \in I \cap (c, c+\delta)$
2. If $f'(c) > 0$, then $\exists \delta > 0$ such that $f(x) < 0$ $\forall x \in I \cap (c-\delta, c)$

<u>*Proof:*</u> ① $\lim_{x \to c}\left(\frac{f(x)-f(c)}{x-c}\right) = f'(c) > 0$ So, $\exists \delta > 0$ such that $x \in I \cap (c, c+\delta)$, then

$$\frac{f(x)-f(c)}{x-c} > 0 \quad \text{but} \quad \underbrace{\left(\frac{f(x)-f(c)}{x-c}\right)}_{>0}\underbrace{(x-c)}_{>0} = f(x) - f(c) \Rightarrow f(x) > f(c)$$

② $\lim_{x \to c}\left(\frac{f(x)-f(c)}{x-c}\right) = f'(c) > 0$ So, $\exists \delta > 0$ such that $x \in I \cap (c-\delta, c)$, then

$\frac{f(x)-f(c)}{x-c} > 0$ However, since $x < c$, we know that $(x-c) < 0$ meaning that

$$\underbrace{\left(\frac{f(x)-f(c)}{x-c}\right)}_{>0}\underbrace{(x-c)}_{<0} = f(x) - f(c) \Rightarrow f(x) - f(c) < 0$$

$$\Rightarrow f(x) < f(c)$$

$\blacksquare$

$\boxed{\text{Theorem}}$: **Darboux's Theorem:** Let $f:[a,b]\to\mathbb{R}$ be differentiable & suppose that $k\in(f'(a),f'(b))$ or $k\in(f'(b),f'(a))$. Then $\exists\, c\in(a,b)$ such that $f'(c)=k$.

Proof: WLOG, assume that $f'(a)<f'(b)$. Then define $g:[a,b]\to\mathbb{R}$ via $x\mapsto kx-f(x)$. Since f is continuous, it follows that g is continuos as well. Hence, the extreme value theorem tells us that g attains an absolute max on $[a,b]$. Since $g'(a)=k-f'(a)>0$. The lemma on the previous page implies that the absolute max of g on $[a,b]$ cannot occur at a nor at b. Therefore the absolute max of g on $[a,b]$ must occur at some $c\in(a,b)\Rightarrow g'(c)=0=k-f'(c)\Rightarrow k=f'(c)$ ∎

6.4: Taylor's Theorem

Title

We define higher-order derivatives in the natural way. That is if $n \in \mathbb{N}$ & $f^{(n-1)}$ is differentiable at c, then,

$$f^{(n)}(c) := (f^{(n-1)})'(c) \quad \text{where } f^{(0)} := f.$$

If f is differentiable at c then we saw with Weierstrass's theorem that f can be approximated effectively by a linear function centered at c:

$$f(x) \approx a_0 + a_1(x-c) := P_1(x)$$

$$\left.\begin{aligned} f(c) &\approx P_1(c) = a_0 \\ f'(c) &\approx P_1'(c) = a_1 \end{aligned}\right\} f(x) \approx f(c) + f'(c)(x-c)$$

If we instead desire a quadratic approximation of f, then we would want f to be twice-differentiable at c.

$$f(x) \approx a_0 + a_1(x-c) + a_2(x-c) = P_2(x)$$

$$\left.\begin{aligned} f(c) &\approx P_2(c) = a_0 \\ f'(c) &\approx P_2'(c) = a_1 \\ f''(c) &\approx P_2''(c) = 2a_2 \end{aligned}\right\} f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$$

We can continue this n-times if f is n-times differentiable at c we can write

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

We call P_n the n-th **Taylor Polynomial** of f at c. We can define the nth residual error term by $R_n := f - P_n$.

On the next page we will prove Taylor's theorem which generalizes the mean value theorem that allows us to approximate differentiable functions with the residual telling us precisely how accurate our approximation is.

Title

Theorem Taylor's Theorem

Let $n \in \mathbb{N}$ & let f be a function on an interval $[a,b]$ such that:

1. $f \in C^n([a,b])$ (meaning f is n-times continuously differentiable on $[a,b]$).
2. f is $(n+1)$-times differentiable on (a,b).
3. let $x_0 \in [a,b]$ be a fixed center point.

Then, $x \in [a,b]$, there exists a scalar c strictly between x & x_0 such that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{}$$

This is the nth Taylor Polynomial $P_n(x)$

This is the Lagrange form of the remainder $R_n(x)$

Proof:

Step 1: Construct our Auxillary Function:

We want to determine the error between the function $f(x)$ & its Taylor Polynomial. Let us define a constant $M \in \mathbb{R}$ representing the normalized error at a specific point x.

Consider a variable t moving between x & x_0. We define the auxillary function $F:[x,x_0] \to \mathbb{R}$ as:

$$F(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k - M(t-x_0)^{n+1}$$

Step 2: Choose M

We need to select the value of the constant M specifically to satisfy the conditions of Rolle's theorem. We need to make $F(x) = F(x_0) = 0$.

at $t = x_0 \Rightarrow F(x_0) = f(x_0) - f(x_0) - 0 = 0$

at $t = x$ we force $F(x) = 0$ by setting

$$M = \frac{1}{(x-x_0)^{n+1}}\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right)$$

This M essentially represents the "error" divided by $(x-x_0)^{n+1}$

Date

Proof: Notice that the summation term in $F(t)$ is the
(Cont...) Taylor polynomial of f centered at x_0. A *vital* property of Taylor polynomials is that their derivatives at the center match the derivatives of f. Therefore, the first n derivatives are equal to zero! That is, $F^{(m)}(x_0) = 0$ for $m \in \{0, 1, \ldots, n\}$

Step 3: Repeated Application of Rolle's Theorem

Since $F(x) = 0 = F(x_0)$ & F is differentiable Rolle's Theorem implies that $\exists c_1 \in (x, x_0)$ such that $F'(c_1) = 0$ Then, since $F'(c_1) = 0 = F'(x_0)$ meaning we can apply Rolle's theorem *again* on this new interval which implies $\exists c_2 \in (c_1, x_0)$ such that $F''(c_2) = 0$. Continuing by this logic, if $F^{(k)}(c_k) = 0$ & $F^{(k)}(x_0) = 0$, then Rolle's theorem implies that $\exists c_{k+1} \in (c_k, x_0)$ such that $F^{(k+1)}(c_{k+1}) = 0$. This implies that after $n+1$ applications we conclude $\exists c_{n+1} \in (c_n, x_0)$ such that $F^{(n+1)}(c_{n+1}) = 0$. Let us call this final value c meaning $F^{(n+1)}(c) = 0$.

Step 4: Compute the derivative

Recall: $F(t) = f(t) - P_n(t) - M(t - x_0)^{n+1}$

↳ Since $P_n(t)$ is a nth degree polynomial meaning that its $(n+1)^{th}$ derivative is zero.

↳ The $M(t - x_0)^{n+1}$ is a $(n+1)^{th}$ degree polynomial

$$\frac{d^{n+1}}{dt^{n+1}}\left(M(t - x_0)^{n+1}\right) = M(n+1)!$$

Which gives us

$$F^{(n+1)}(t) = f^{(n+1)}(t) - 0 - M(n+1)!$$

evaluating at the specific point c where we know $F^{(n+1)}(c) = 0$, we get $0 = f^{(n+1)}(c) - M(n+1)!$

Solving this for M yields $M = \dfrac{f^{(n+1)}(c)}{(n+1)!}$ Finally, we substitute this value M back into our original expression for $f(x)$ gives us:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$\boxed{\text{Example: Applying Taylor's Theorem}}$: Find the Taylor polynomial of the function $f(x) = (1+x)^{1/2}$ about 0.

We begin bus taking the first few derivatives to give us an idea of the pattern that theys follow.

$$f'(x) = \frac{1}{3}(1+x)^{-2/3} \quad \& \quad f''(x) = -\frac{2}{9}(1+x)^{-5/3} \quad \& \quad f'''(x) = \frac{10}{27}(1+x)^{-8/3}$$

Hence, we can write.

$$f(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}(1+c)^{-8/3}x^3$$

for some $c \in (0, x)$ (or $c \in (x, 0)$)

If we want to approximate the value of $(1.3)^{1/3}$. Well,

$$(1.3)^{1/3} = f\left(\frac{3}{10}\right) = 1 + \frac{1}{10} - \frac{1}{100} + \frac{5}{81}(1+c)^{-8/3}\left(\frac{3}{10}\right)^3$$

for some $c \in (0, 3/10)$

$$\Rightarrow \left| f\left(\frac{3}{10}\right) - 1 - \frac{1}{10} + \frac{1}{100} \right| = \left| \frac{5}{81}(1+c)^{-8/3}\left(\frac{3}{10}\right)^3 \right|$$

Then, since $0 < (1+c)^{-8/3} < 1$

$$\leq \frac{5}{3(10^3)} = \frac{1}{600} < \frac{1}{100}$$

$$\therefore f\left(\frac{3}{10}\right) \approx 1 + \frac{1}{10} - \frac{1}{100} = 1.09$$

is accurate up to two decimal places!

6.4 Taylor's Theorem

$\boxed{\text{Theorem}}$: Let I be an interval with an interior point x_0. Suppose that $n \geq 2$ & that $f \in \mathcal{C}^n(I)$ such that $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ but $f^{(n)}(x_0) \neq 0$. Then,

1. If n is even & $f^{(n)}(x_0) > 0 \Rightarrow f$ has a relative min. at x_0.
2. If n is even & $f^{(n)}(x_0) < 0 \Rightarrow f$ has a relative max. at x_0.
3. If n is odd $\Rightarrow f$ has neither a relative min. nor a relative max. at x_0.

Proof: Taylor's Theorem $\Rightarrow f(x) = P_{n-1}(x) + \dfrac{f^{(n)}(c)}{n!}(x-x_0)^n$ for some $c \in (x_0, x)$ or (x, x_0)

But $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0 \Rightarrow P_{n-1}(x) = f(x_0)$

$$\Rightarrow f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

Since $f^{(n)}$ is continuous on I, $f^{(n)}(x_0) \neq 0$, & x_0 is an interior point of I, there must exist some $\delta > 0$ such that

$x \in (x_0 - \delta, x_0 + \delta) \subseteq I \Rightarrow f^{(n)}(x) > 0$ if $f^{(n)}(x_0) > 0$ &
$f^{(n)}(x) < 0$ if $f^{(n)}(x_0) < 0$

$\boxed{\hookrightarrow \text{set } \varepsilon = \dfrac{|f^{(n)}(x_0)|}{2} \text{ \& find } \delta \text{ for this } \varepsilon.}$

Thus, if $f^{(n)}(x_0) > 0$ & n is even

$$\Rightarrow f(x) = f(x_0) + \underbrace{\frac{f^{(n)}(c)}{n!}}_{>0}\underbrace{(x-x_0)^n}_{>n} > f(x_0)$$

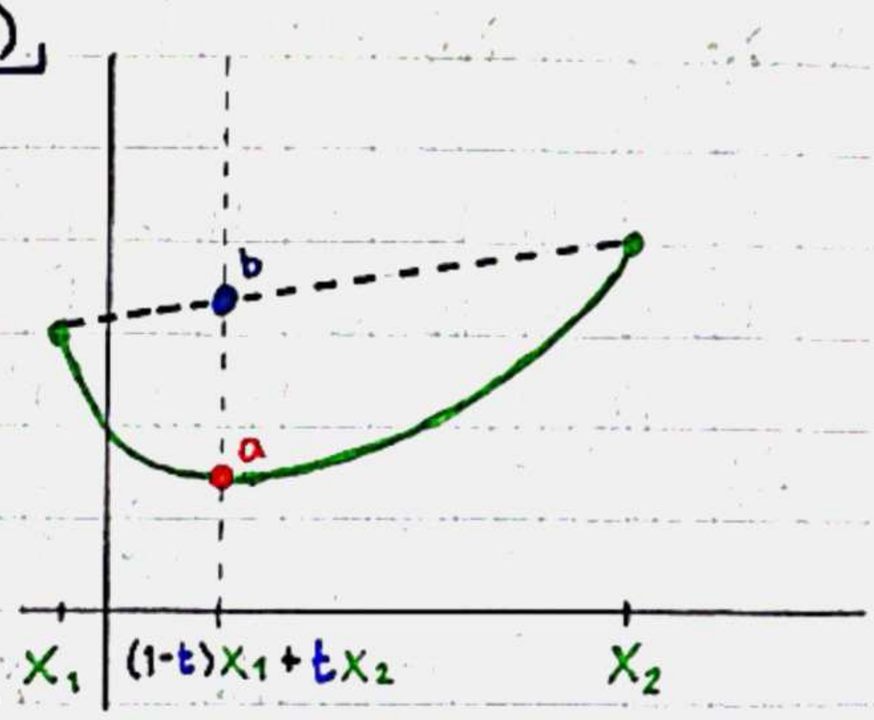
$\therefore f$ has a relative min. at x_0. A similar argument proves ②, but if n is odd, then $(x-x_0)^n > 0$ if $x > x_0$ but $(x-x_0)^n < 0$ if $x < x_0$ making $\dfrac{f^{(n)}(c)}{n!}(x-x_0)^n$ flips signs at x_0 meaning $f(x) - f(x_0)$ does too meaning f cannot have a relative extremeum at x_0. ∎

6.4 Taylor's Theorem

Let I be an interval. $f: I \to \mathbb{R}$ is **convex** on I if $\forall t \in [0,1]$ & $x_1, x_n \in I$ we have:

$$\underbrace{f((1-t)x_1 + tx_2)}_{a} \leq \underbrace{(1-t)f(x_1) + tf(x_2)}_{b}$$

That is, the line segment between $(x_1, f(x_1))$ & $(x_2, f(x_2))$ lies <u>above</u> the graph of f. (in calculus we called convex functions *concave up*.)



$\boxed{\text{Theorem}}$: Let I be an open interval, & Suppose that f is twice differentiable on I. Then f is convex on I If & only if $f''(x) \geq 0$ $\forall x \in I$.

Proof: ($\Rightarrow$) f is convex on $I \Rightarrow$ the slopes of the secant lines on $\{(x, f(x)) : x \in I\}$ are increasing. If we let $x < y < z$ such that $x, y, z \in I$. Since the function $g: [0,1] \to [x, z]$ via $s \mapsto xs + (1-s)z$ is a bijection, there is a unique $t \in [0,1]$ such that $g(t) = y$. Furthermore, f is convex & so $f(\underbrace{g(t)}_{y}) \leq f(x)t + (1-t)f(z)$. The precise $t \in [0,1]$ such that $g(t) = y$ is

$$f(y) \leq f(x)\left(\frac{y-z}{x-z}\right) + \left(\frac{x-z-y+z}{x-z}\right)f(z) \Leftarrow t := \frac{y-z}{x-z}$$

$$= f(x)\left(\frac{y-z}{x-z}\right) + \left(\frac{x-y}{x-z}\right)f(z) = f(x) + \left(\frac{f(z)-f(x)}{z-x}\right)(y-x)$$

$$\Rightarrow \frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}$$ As desired. However, this means the difference quotients of f on I are increasing, by the order theorem for functional limits we have that f' is increasing on I meaning f'' must be nonnegative on I by Fermat's theorem.

(continued on next page...)

Proof: ($\Leftarrow$) Let $x, z \in I$ & let $t \in [0,1]$ & let $y \in [x,z]$ be
(cont...) defined by $y := (1-t)x + tz$ (Assume $x \leq z$)
Taylor's Theorem $\Rightarrow \exists c_1 \in [x,y]$ such that

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(c_1)(y-x)^2$$

Similarly,
Taylor's Theorem $\Rightarrow \exists c_2 \in [y,z]$ such that

$$f(z) = f(y) + f'(y)(z-y) + \frac{1}{2}f''(c_2)(z-y)^2$$

Since f'' is non-negative on I, we have

$$(1-t)f(x) + tf(z) =$$
$$= f(y) + f'(y)\underbrace{((1-t)x + tz - y)}_{=0} + \underbrace{\tfrac{1}{2}(1-t)f''(c_1)(x-y)^2 + \tfrac{1}{2}tf''(c_2)(z-y)^2}_{\geq 0}$$
$$\geq f(y) = f((1-t)x + tz)$$

$\therefore$ f is convex on I. ∎

Newton's Method: Let f be differentiable & suppose $f(r) = 0$ & that x_1 is an initial estimate of r. Then, the line tangent to the graph of f at x_1 is given by L_1 which is defined by $y - f(x_1) = f'(x_1)(x - x_1)$. L_1 crosses the x-axis at some x_2. In particular, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. We can then consider the line L_2 defined by $y - f(x_2) = f'(x_2)(x - x_2)$. We can continue this process iteratively & it will turn out that the sequence (x_n) will converge to a root of f rapidly, provided that some conditions are satisfied. We will rigorously explore this procedure (& its limitations) on the following page(s).

$\boxed{\text{Theorem}}$: **Newton's Method** Let $f:[a,b]\to\mathbb{R}$ be twice-differentiable on $[a,b]$ & suppose that

1. $f(a)\cdot f(b)<0$
2. $\exists m>0$ such that $|f'(x)|\geq m \quad \forall x\in[a,b]$
3. $\exists M>0$ such that $|f''(x)|\leq M \quad \forall x\in[a,b]$

Then we define the convergence constant $K=\frac{M}{2m}$

Then, there exists a subinterval $I^*\subseteq[a,b]$ containing a unique root $r\in f^{-1}(\{0\})$ such that if $x_1\in I^*$ is arbitrary & we define the *Newton Iteration*

$x_{n+1}:=x_n-\frac{f(x_n)}{f'(x_n)}$ then the sequence (x_n) remains in I^* for all $n\in\mathbb{N}$, $\lim_{n\to\infty}(x_n)=r$

& $|x_{n+1}-r|\leq K|x_n-r|^2 \quad \forall n\in\mathbb{N}$.

<u>*Proof*</u>: Step 1: Existence & Uniqueness of the Root

$$f(a)\cdot f(b)<0 \Rightarrow \begin{cases} f(a)>0 \ \& \ f(b)<0 \\ f(a)<0 \ \& \ f(b)>0 \end{cases} \ \underline{\text{or}}$$

By the intermediate value theorem there exists at least one $r\in(a,b)$ with $f(r)=0$.
That is, $f^{-1}(\{0\})\cap[a,b]\neq\emptyset$
Suppose for a contradiction that $r,s\in[a,b]\cap f^{-1}(\{0\})$ with $r<s$. By Rolles Theorem $\exists c\in(r,s)$ such that $f'(c)=0$, but this contradicts our second condition ②. Hence, the root r is unique!

Step 2: Deriving the Error Recurrence via Taylor's Theorem

Let $x'\in[a,b]\setminus\{r\}$. By Taylor's Theorem there exists $c'\in(x',r)$ such that:

$$f(r)=f(x')+f'(x')(r-x')+\tfrac{1}{2}f''(c')(r-x')^2$$

Since $f(r)=0$ we can solve for $-f(x')$

$$-f(x')=f'(x')(r-x')+\tfrac{1}{2}f''(c')(r-x')^2$$

With this we can define the Newton Iterate

$x'':=x'-\frac{f(x')}{f'(x')}$ $\Rightarrow$ substituting in our value of $f(x')$ gives us:

$$x''=x'+\frac{f'(x')(r-x')+\frac{1}{2}f''(c')(r-x')^2}{f'(x')}$$

$$x''=x'+(r-x')+\frac{f''(c')}{2f'(x')}(r-x')^2=r+\frac{f''(c')}{2f'(x')}(r-x')^2$$

$$\therefore x''-r=\frac{f''(c')}{2f'(x')}(r-x')^2$$

(continued on the next page...)

Proof: (cont...) Step 3: Establish Quadratic Error Bound

$$|x''-r| = \left|\frac{f''(c')}{2f'(x')}\right| |x'-r|^2 \leq \frac{M}{2m}|x'-r|^2 = K|x'-r|^2$$

Step 4: Construct the invariant Interval I^*

Choose $\delta > 0$ such that $\delta < 1/K$, then define $I^* := [r-\delta, r+\delta] \subseteq [a,b]$. Then, $x_n \in I^* \Rightarrow |x_n - r| \leq \delta$. Using the error bound from step 3:

$$|x_{n+1}-r| \leq K|x_n-r|^2 \leq K\delta^2 < K\frac{1}{K}\delta = \delta$$

Thus $x_n \in I^*$ $\forall n \in \mathbb{N}$ whenever $x_1 \in I^*$

Step 5: Prove convergence.

For $x_1 \in I^*$, we iterate the bound

$|x_{n+1}-r| \leq K|x_n-r|^2 \leq K\delta|x_n-r| \Rightarrow$ let $\lambda := K\delta < 1$

$|x_{n+1}-r| \leq \lambda|x_n-r| \leq \lambda^2|x_{n-1}-r| \leq \cdots \leq \lambda^n|x_1-r|$

Since $\lambda < 1$, $\lim(\lambda^n) = 0$ (as $n \to \infty$)

$$\therefore \lim_{n\to\infty}(|x_n-r|) = 0 \Rightarrow \lim_{n\to\infty}(x_n) = r$$

NOTE! The three conditions of Newton's method need to be checked each time you apply it. Otherwise, you can run into problems, like the sequence (x_n) oscilating between two points
↳ $(x_1\ x_2\ x_1\ x_2\ x_1\ x_2 \ldots)$ no convergence ☹

Title

6.4 Taylor's Theorem

$\boxed{\text{Example: Newton's Method}}$ Let $f(x) = x^2 - 2$. Then f is twice differentiable on $\mathbb{R}$, & hence, on any closed sub-interval of $\mathbb{R} \Rightarrow$ $f'(x) = 2x$ $\forall x \in \mathbb{R}$
$f''(x) = 2$ Observe that
$f(1) = 1-2 < 0 < f(3) = 9-2 \Rightarrow$ The intermediate value theorem says $\exists r \in (1,3)$ such that $f(r) = 0$. Furthermore, $\underbrace{|f'(x)| \geq 2 > 0}_{\text{for all } x \in [1,3]}$ & $|f''(x)| = 2$, so we can set $M = m = 2$

Following the procedure of Newton's Method, let $K := \frac{M}{2m} = \frac{2}{4} = \frac{1}{2}$ Next, we need to find a $\delta < 1/K = 2$ such that $I^* := [r-\delta, r+\delta] \subseteq [1,3]$
Since $f(5/4) < 0 < f(3/2)$, $\delta = 1/4$ will suffice. Now, let $x_1 = 5/4$, meaning $x_1 \in I^*$ since $r - 1/4 > 1$ & $r + 1/4 < 3/2 + 1/4 = 7/4$, & since $f(5/4) \neq 0$, $r \neq 5/4$
Then, by Newton's Method $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ meaning,
$x_2 = \frac{5}{4} - \frac{(5/4)^2 - 2}{2(5/4)} = \frac{57}{40}$ but, $|x_2 - r| \leq 2|x_1 - r|^2 \leq 1/8$,
So we still are not within one decimal place of accurracy of $r = \sqrt{2}$. But, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{6449}{4560}$
$|x_3 - r| < (2 - \frac{1}{4})^2 |x_1 - r| \leq \frac{1}{16} < \frac{1}{10}$
In general, $|x_{n+1} - r| < \frac{1}{2^n} \frac{1}{4} = \frac{1}{2^{n+2}}$.

Date